7

Time Value of Money: Advanced Concepts and Applications

Learning Objectives

An understanding of the material in this chapter should enable the student to

7-1. Calculate the present value of a series of uneven cash flows.

7-2. Calculate the present value of a series of cash flows that grow by a constant percentage.

7-3. Calculate the future value of a series of uneven cash flows.

7-4. Calculate the future value of a series of cash flows that grow by a constant percentage.

7-5. Compute the net present value (NPV) of an investment project.

7-6. Compute the internal rate of return (IRR) on an investment project.

7-7. Solve several types of time-value problems when compounding or discounting occurs more frequently than annually.

DEALING WITH UNEVEN CASH FLOWS

This chapter describes calculations for the present value and future value of a stream of uneven or irregular payments on an annual basis. You will see that these calculations are useful for problems that financial advisors frequently encounter, such as college funding or evaluating investments.

Present Value of Uneven Cash Flows

Assume, for example, that a young man will be entering college in one year. The estimated tuition is $6,000 to be paid at the start of the freshman year, $6,700 at the start of the sophomore year, $7,300 at the start of the junior year, and $8,200 at the start of the senior year. How much would a client need to set aside today in an account earning 8.5 percent compound interest in order to pay the four tuition payments as they come due? In other words, what is the present value of this series of uneven cash flows at an 8.5 percent discount rate?

Using the PVSS and the PVA Formulas to Compute the Present Value of Uneven Cash Flows

As described in the previous chapter, the formula for calculating the present value of each of the future tuition payments is the present value of a single sum (PVSS) formula with the appropriate PVSS factor. The PVSS formula is

$$\text{PVSS} = \text{FVSS} \times [1 \div (1 + i)^n]$$

Because the bracketed portion of the equation is the PVSS factor, the PVSS formula can be written as

$$\text{PVSS} = \text{FVSS} \times \text{PVSS factor}$$

The solution to the tuition problem using the PVSS formula with the appropriate PVSS factor is actually a five-step process. The first four steps involve finding the present value of each year's tuition payment. Step five, which provides the answer, requires you to sum the present values computed in the first four steps.

Table 7-1
Solving the Tuition Problem

Step 1 (year 1 tuition):	$\$6,000 \times [1 \div (1 + .085)^1] = \$6,000 \times .9217 = \$5,530.20$
Step 2 (year 2 tuition):	$\$6,700 \times [1 \div (1 + .085)^2] = \$6,700 \times .8495 = \$5,691.65$
Step 3 (year 3 tuition):	$\$7,300 \times [1 \div (1 + .085)^3] = \$7,300 \times .7829 = \$5,715.17$
Step 4 (year 4 tuition):	$\$8,200 \times [1 \div (1 + .085)^4] = \$8,200 \times .7216 = \$5,917.12$
Step 5 (add years 1–4):	$22,854.14

If $22,854.14 is deposited today at 8.5 percent compound interest, the account will be sufficient to pay each of the estimated tuition payments as it falls due over the next 4 years. At the end of the 4-year period, the account will be depleted, as illustrated below.

Table 7-2
Liquidation of a Capital Sum Compounding at 8.5 Percent Interest Through Uneven Cash Withdrawals

Year	Beginning Balance	Interest Earnings	Cash Withdrawal	Ending Balance
1	$22,854.14	$1,942.60	$6,000	$18,796.74
2	18,796.74	1,597.72	6,700	13,694.46
3	13,694.46	1,164.03	7,300	7,558.49
4	7,558.49	642.47	8,200	0.96

ungrouped cash flows

grouped cash flows

This example involving tuition payments illustrates *ungrouped cash flows,* which are a series of cash flows where there are no consecutive payments of the same amount and direction, that is, the cash flow can be either positive (inflow) or negative (outflow). In ungrouped cash flows, each payment must be discounted separately. In situations with *grouped cash flows,* where some of the consecutive payments are of the same amount and flow in the same direction (that is, either as cash inflows or outflows), a shortcut can be used to find the present value.

A corporate bond provides a good illustration of an uneven cash flow where some cash flows are grouped. For example, consider a bond that will pay you interest of $80 per year at the end of each of the next 6 years, as well as the $1,000 face amount at the end of the sixth year.[44] If you want to earn a 13 percent rate of return on this investment, how much would you be willing to pay for the bond? That is, determine the present value, discounted at 13 percent, of the bond's cash flows as depicted in *Figure 7-1, Time Line Depiction of a Bond Value*.

This series of cash flows actually consists of two separate parts. Part one is an annuity of $80 per year for 5 years (or an annuity of $80 per year for 6 years), and part two is a single sum of $1,080 at the end of the sixth year (or a single sum of $1,000 at the end of the sixth year). To compute the present value of the entire series of cash flows, first calculate the present value of the annuity (PVA). Next calculate the PVSS. Finally add the results of the two parts.

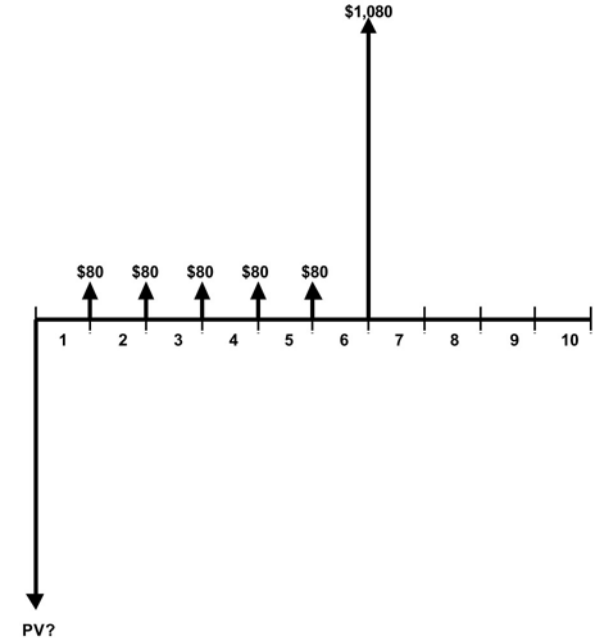
The solution for part one can be calculated using the PVA formula. This formula is

$$\text{PVA} = \text{annual payment} \times \{1 - [1 \div (1 + i)^n] \div i\}$$

The bracketed portion of the equation is the PVA factor.

44. Most corporate bonds pay interest semiannually. For purposes of simplicity, however, annual payments are assumed here. A later section of this chapter deals with compounding and discounting where cash flows occur more frequently than once per year.

Figure 7–1 Time Line Depiction of a Bond Value



This time line depicts the uneven stream of cash flows provided by a bond-five annual payments of $80 each and a sixth of $1,080. The problem is to compute the present value of the stream of cash flows.

Because the bracketed portion is the PVA factor, the PVA formula can be written as

$$\text{PVA} = \text{annual payment} \times \text{PVA factor}$$

The use of the PVA formula with the appropriate PVA factor to find the present value of the series of $80 interest payments is as follows:

$$\begin{aligned}\text{PVA} &= \$80 \times \{1 - [1 \div (1.13)^5] \div .13\} \\ &= \$80 \times 3.5172 \\ &= \$281.38\end{aligned}$$

The part two solution can be calculated using the PVSS formula. This formula, as previously described, is

$$\begin{aligned}\text{PVSS} &= \text{FVSS} \times \text{PVSS factor} \\ &= \$1{,}080 \times .4803 \\ &= \$518.72\end{aligned}$$

Thus, the present value of the entire series of cash flows, that is, the price you would be willing to pay for the bond, is $800.10 ($281.38 + $518.72). If the bond currently sells for this amount or less, you would buy the bond.

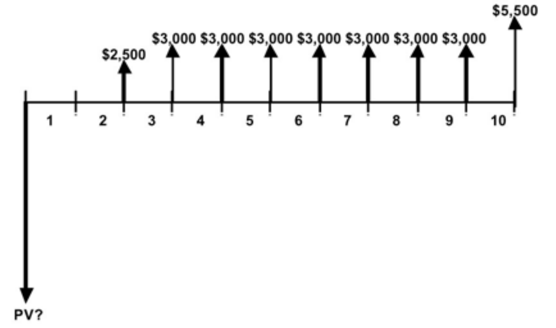
In another example involving grouped cash flows, assume you loan a simple sum to a borrower today. The repayment schedule calls for the borrower to make payments to you as follows:

End of Year	Amount of Payment
1	$ 0
2	2,500
3–9	3,000
10	5,500

Assuming an 11 percent interest rate, determine the present value of this series of payments, which is depicted in *Figure 7-2, Time Line Depiction of the Present Value of Uneven Cash Flows*

To solve this problem without the aid of your HP-10BII requires several steps. First, find the present value of the $2,500 payment at the end of year 2. Second, calculate the present value of a 7-year, $3,000 annuity. Third, discount the present value of that annuity as a single sum to its present value today, rather than its value 2 years from now when the annuity begins. Refer to the *Figure 7-2, Time Line Depiction of the Present Value of Uneven Cash Flows* time line depiction and note that this regular annuity begins at the start of the third year (the end of the second year) with the first $3,000 payment to be made at the end of the third year. Fourth, find the present value of the $5,500 payment at the end of year 10. Finally, sum the PVSS of the first payment, the twice-discounted value of the PVA of the next seven payments, and the PVSS of the ninth payment to obtain the present value of the entire series of cash flows.

Figure 7–2 Time Line Depiction of the Present Value of Uneven Cash Flows



This time line depicts a problem in which the task is to compute the present value of a stream of cash flows consisting of zero at the end of year 1, $2,500 at the end of year 2, $3,000 at the end of years 3-9, and $5,500 at the end of year 10.

Using the PVSS and the PVA formulas with the appropriate factor tables to explain each of the steps and calculate all the values, the problem is solved as follows:

In step 1, calculate the PVSS for the $2,500 payment at the end of year 2.

$$\begin{aligned} \text{PVSS} &= \text{FVSS} \times [1 \div (1 + .11)^2] \\ &= \text{FVSS} \times \text{PVSS factor} \\ &= \$2{,}500 \times .8116 \\ &= \$2{,}029.00 \end{aligned}$$

In step 2, calculate the PVA of 7 years for the $3,000 payments at the end of years 3 through 9.

$$\begin{aligned} \text{PVA} &= \text{annual payment} \times \{1 - [1 \div (1 + .11)^7] \div .11\} \\ &= \text{annual payment} \times \text{PVA factor} \\ &= \$3{,}000 \times 4.7122 \\ &= \$14{,}136.60 \end{aligned}$$

In step 3, calculate the PVSS for the PVA (determined in step 2) of $14,136.60.

$$\begin{aligned} \text{PVSS} &= \text{FVSS} \times [1 \div (1 + .11)^3] \\ &= \text{FVSS} \times \text{PVSS factor} \\ &= \$14{,}136.60 \times .07312 \\ &= \$10{,}336.68 \end{aligned}$$

In step 4, calculate the PVSS for the $5,500 payment at the end of year 10.

$$\begin{aligned} \text{PVSS} &= \text{FVSS} \times [1 \div (1 + .11)^{10}] \\ &= \text{FVSS} \times \text{PVSS factor} \\ &= \$5{,}500 \times .3522 \\ &= \$1{,}937.10 \end{aligned}$$

In step 5, total the present values of the cash flows from steps 1, 3, and 4.

$ 2,029.00	(PVSS from step 1)
10,336.68	(PVSS from step 3)
1,937.10	(PVSS from step 4)
$14,302.78	

So far, all of the cash flows in the previous problems have been inflows and thus positive. In some problems, however, one or more of the cash flow amounts may be an outflow and thus negative. For example, assume you own rental property that is expected to generate net income of $15,000 per year at the end of each of the next 15 years, except for year 10. In year 10 you anticipate replacing the roof and making various other repairs, resulting in a net cash outflow for the year of $10,000. What is the present value of this stream of cash flows discounted at 9 percent? To solve this problem, like the last problem, requires several steps.

The solution to the problem can be found using the PVA and the PVSS formulas with the appropriate factor tables to explain each of the steps and calculate all the values. Note that in step 2 a negative sign representing the cash outflow must be used.

In step 1, calculate the PVA of 9 years for the $15,000 cash inflows that begin at the end of year 1 and continue to the end of year 9.

$$\begin{aligned} \text{PVA} &= \text{annual payment} \times \{1 - [1 \div (1 + .09)^{9}] \div .09\} \\ &= \text{annual payment} \times \text{PVA factor} \\ &= \$15{,}000 \times 5.9952 \\ &= \$89{,}928.00 \end{aligned}$$

In step 2, calculate the PVSS of the $10,000 cash outflow at the end of year 10.

$$\begin{aligned} \text{PVSS} &= \text{FVSS} \times [1 \div (1 + .09)^{10}] \\ &= \text{FVSS} \times \text{PVSS factor} \\ &= -\$10{,}000 \times .4224 \\ &= -\ \$4{,}224.00 \end{aligned}$$

In step 3, calculate the PVA of 5 years for the $15,000 cash inflows that begin at the end of year 11 and continue to the end of year 15.

$$\begin{aligned} \text{PVA} &= \text{annual payment} \times \{1 - [1 \div (1 + .09)^{5}] \div .09\} \\ &= \text{annual payment} \times \text{PVA factor} \\ &= \$15{,}000 \times 3.8897 \\ &= \$58{,}345.50 \end{aligned}$$

In step 4, calculate the PVSS of the PVA (determined in step 3) of $58,345.50.

$$\begin{aligned} \text{PVSS} &= \text{FVSS} \times [1 \div (1 + .09)^{10}] \\ &= \text{FVSS} \times \text{PVSS factor} \\ &= \$58{,}345.50 \times .4224 \\ &= \$24{,}645.14 \end{aligned}$$

In step 5, total the present values of the cash flows from steps 1, 2, and 4.

$ 89,928.00	(PVA from step 1)
-4,224.00	(PVSS from step 2)
24,645.14	(PVSS from step 4)
$110,349.14	

This problem could be solved in several other ways. One approach, for instance, is to calculate the PVA of 15 years for the $15,000 cash inflows. Then you could subtract the PVSS of the outflow of $25,000 at the end of year 10 from the PVA. Remember you received a $15,000 cash inflow as rental income at the end of year 10, which was offset by the $25,000 cash outflow for repairs at the end of year 10, leaving a $10,000 net cash outflow for year 10.

Using an HP-10BII to Solve Present Value Problems With Uneven Cash Flows

The preceding problems illustrated simple patterns of payment and were solved using time-value formulas. Although not technically difficult, the process of solving these problems using time-value formulas is tedious and cumbersome, even with the aid of factor tables. To speed up the process, you could use an HP-10BII to calculate the present value of each cash flow or group of cash flows. However, even with the aid of your HP-10BII, the process is still rather tedious and cumbersome unless you learn how to use some new keys that can speed up the procedure. With the knowledge of these new keys, your HP-10BII can be especially helpful when there is a lengthy series of cash flows with many different payment amounts, including positive (inflow), zero, and negative (outflow) amounts. If a problem exceeds the capacity of your HP-10BII, you can use a computer to solve it.

Speaking of the HP-10BII's capacity, it has the capability to solve problems with up to 14 end-of-year or 15 beginning-of-year cash flows when there are no consecutive equal cash flows. When there are consecutive equal cash flows that can be grouped, the HP-10BII's capacity is expanded.

As suggested above, to efficiently solve uneven cash flow problems with your HP-10BII, you must learn how to use several new keys. First is the CF_j key in the third row of the keyboard. This key is used to record the items in a series of uneven cash flows, starting at the beginning of the time horizon. The j in CF_j refers to the timing of the cash flow. The cash flow at the start of year one is CF_o, and the subsequent cash flows (or groups of cash flows) are numbered consecutively from Cf_1 through as many as Cf_{14}.

To compute the present value of a series of uneven cash flows, start by entering all of the cash flows in sequence. Then enter the discount rate using the I/YR key as usual. Next, take notice of the NPV (net present value) key in the second row of the keyboard. Together the  and NPV keys are used to produce the solution. Also, as noted above, the CF_j key assumes that payments begin immediately, so it makes no difference whether your HP-10BII is set for beginning-of-year or end-of-year payments. If the first payment in the problem begins at the end of the year, you must enter 0 as the first cash flow to tell your HP-10BII that there is no cash flow at the beginning of the time horizon.

To illustrate the process, assume that you have a deferred compensation agreement with your employer. Under the agreement the employer is obligated to pay you the following amounts:

End of Year	Amount of Payment
1	$ 0
2	70,000
3	80,000
4	85,000
5	95,000

Using a 12.5 percent discount rate, determine the present value of this income stream. After clearing your HP-10BII's memory (, C ALL), press the following keys: 0 and CF_j (because there is no immediate cash flow at the start of year one); 0 and CF_j (because there is no immediate cash flow at the end of year one); 70000 and CF_j (to enter the cash flow at the end of year two); 80000 and CF_j (to enter the cash flow at the end of year three); 85000 and CF_j (to enter the cash flow at the end of year four); 95000 and CF_j (to enter the cash flow at the end of year five). Next press 12.5 and I/YR (to enter the discount rate). Finally, to solve the problem, press and NPV, which provides the answer of $217,278.53.

HP-10BII: Keystrokes for Computing the Present Value of a Series of Uneven (End-of-Year) Cash Flows

Keystrokes	***Explanation***
, C ALL	clearing memory
0, CF_j	no payment at beginning of first year
0, CF_j	no payment at end of first year
70000,CF_j	end of second year payment
80000, CF_j	end of third year payment
8 5000, CF_j	end of fourth year payment
95000, CF_j	end of fifth year payment
12.5, I/YR	discount interest rate
, NPV	217,278.53 displayed

Solving Problems That Include Grouped Cash Flows. Your HP-10BII can solve present value problems with more than 14 end-of-year or 15 beginning-of-year cash flows if the series includes some grouped data. Grouped data counts as only one cash flow amount. To illustrate, a cash flow of $50,000 followed by 40 annual cash flows of $1,000 involves 41 cash flows but only two cash flow entries for the HP-10BII. For problems that include grouped data, enter the cash flow amount, CF_j, the number of times the cash flow occurs in succession (including the first time), , and N_J (the secondary function on the CF_j key in the third row).

For example, consider an investment that generates a cash inflow of $6,000 per year for 5 years, starting at the end of year one, followed by $4,000 per year for 10 years, followed by $1,000 per year for 15 years, followed by a lump-sum payment of $20,000. Determine the present value of this income stream discounted at 8 percent.

Using your HP-10BII, press and C ALL to clear its memory. Then press 0 and CF_j (because there is no cash flow at the start of year one). Then press 6000, CF_j, 5 (because $6,000 will be paid 5 consecutive years), , and N_j. Next enter 4000, CF_j, 10 (because $4,000 will be paid 10 consecutive years), , and N_j. Then enter 1000, CF_j, 15 (because $1,000 will be paid 15 consecutive years), , and N_j. Next enter 20000 and CF_j (for the final one year cash flow). Then enter 8 and I/YR (for the discount rate). Finally, press and NPV to compute the answer, which is $46,761.96. If you would like to discount this same cash flow again, but at a different rate, simply enter the new rate and I/YR. Then press and NPV to compute the answer at the new discount rate.

HP-10BII: Keystrokes for Computing the Present Value of Grouped Cash Flows

Keystrokes	***Explanation***
, C ALL	clearing memory
0, CF_j	no payment at the beginning of year one
6000, CF_j, 5, , N_j	5 consecutive $5,000 cash flows
4000,CF_j, 10, , N_j	10 consecutive $4,000 cash flows
1000, CF_j, 15, , N_j	15 consecutive $1,000 cash flows
20000, CF_j	one $20,000 cash flow
8, I/YR	discount interest rate
, NPV	46,761.96 displayed

Cash Flows at Beginning of Year. In the preceding example, cash flows began at the end of the first year. If a cash flow occurs at the beginning of the first year, only a slight change in procedure is needed to find the present value of the cash flows. Enter the amount of the initial cash flow (if it occurs only once as ungrouped data) and CF_j. Then proceed in the usual manner to enter the size and number of each of the remaining cash flows. However, if the first cash flow consists of grouped data—for example, $1,000 at the beginning of years 1, 2, 3, and 4—the data must be treated as two separate cash flows, $1,000 occurring at the start of the first year followed by a group of three additional cash flows of $1,000 each.

For example, at a discount rate of 12 percent, what is the present value of the following series of cash flows?

Beginning of Year	Amount of Cash Flow
1–5	$3,500
6–10	2,500
11	1,500
12	1,000

First clear the HP-10BII's memory (, C ALL) and then enter the first five cash flows by pressing 3500 and CF_j (for the first cash flow); 3500, CF_j, 4, , and N_j (for the remaining four $3,500 cash flows). Next enter the next five cash flows (for years 6 through 10) by pressing 2500, CF_j, 5, , and N_j. Then enter the last two cash flows by pressing 1500 and CF_j (for year 11); 1000 and CF_j (for year 12). Next enter the discount rate by pressing 12 and I/YR. Finally enter and NPV to compute the answer, which is $20,628.41.

HP-10BII: Keystrokes for Computing Present Value If There Is a Cash Flow at the Beginning of the First Period	
Keystrokes	***Explanation***
, C ALL	clearing memory
3500, CF_j	cash flow at beginning of first year
3500, CF_j, 4, , N_j	4 consecutive cash flows for years 2 through 5
2500, CF_j, 5, , N_j	5 consecutive cash flows for years 6 through 10
1500, CF_j	the cash flow for year 11
1000, CF_j	the final cash flow for year 12
12, I/YR	discount interest rate
, NPV	20,628.41 displayed

Payments or Deposits Growing by a Constant Percentage. In many cases, financial advisors need to discount a stream of payments that increase each year by a constant percentage. Educational funding is a frequent application, when an assumed education inflation rate is applied to the tuition payments.

To illustrate this concept in another application, assume that you wish to have an income stream of $30,000 per year in constant purchasing power for the next 10 years, with the first payment occurring immediately. In other words, you wish to receive a total of 10 payments in the form of an annuity due. To fund this annuity, how much would you have to set aside today if the annual inflation rate is expected to be 5 percent and the deposit will earn a compound annual interest rate of 8 percent per year? To answer this question, you need to determine the present value, discounted at 8 percent, of a 10-year stream of payments beginning with $30,000 and increasing by 5 percent per year.

One approach to solving this type of problem is to use a formula that takes inflation into consideration. However, rather than attempting to solve this problem by using a formula, it is much easier and quicker to use your HP-10BII. First set it for beginning-of-year payments and clear its memory. Then enter 30000 and PMT (for the first payment due immediately). Next enter 1.08, ÷, 1.05,-, 1, ×, 100, =, and I/YR (for the inflation-adjusted interest rate). Then enter 10 and N (for the number of payments). Finally press PV to compute the answer, which is $265,147.15.

If you want to see the answer under an end-of-year assumption without reentering all the information, divide the beginning-of-year answer by 1.08. In other words, enter PV on your HP-10BII and the beginning-of-year answer, $265,147.15, reappears. Then enter ÷, 1.08, and = to compute the end-of-year answer, which is $245,506.62.

HP-10BII: Keystrokes for Computing the Present Value of Payments or Deposits That Grow at a Constant Rate If the First Payment Is Due Immediately	
Keystrokes	***Explanation***
, C ALL	clearing memory
, BEG/END	only if BEGIN not displayed
30000, PMT	first payment
1.08, ÷, 1.05, -, 1, ×, 100, =, I/YR	adjusts the discount rate (8%) by the inflation rate (5%) for an adjusted rate of 2.86
10, N	number of payments
PV	-265,147.15 displayed
Without Reentering All the Information If the First Payment Is Due One Year From Now	
Keystrokes	***Explanation***
PV	-265,147.15 displayed
÷, 1.08, =	-245.506.62 displayed

Alternatively you can start over.

To obtain the adjusted discount rate, you may be tempted to simply subtract the inflation rate of 5 percent from the discount rate of 8 percent. Unfortunately, this simple approach is incorrect. To understand why, assume that the inflation rate, as before, is expected to be 5 percent during the year and that you want to earn an inflation-adjusted rate of 8 percent on a $100 investment in order to have $108 of purchasing power ($100 × 1.08) in today's dollars one year from now. If inflation is 5 percent during the year, how much will you need at the end of one year to have the equivalent of $108 in today's dollars? The answer is $113.40, which is $108 × 1.05, where 1.05 is the adjustment factor for inflation. In other words, you must earn a nominal rate of 13.4 percent [($113.40 – 100) ÷ $100] to earn a real (inflation-adjusted) rate of 8 percent if inflation is 5 percent. (To obtain 13.4 percent instead of just 13 percent on the display screen, you need to change the decimal precision on your HP-10BII from two to four. This is done by entering , DISP, and 4 for the number of decimal places to be displayed.

HP-10BII: Keystrokes for Computing the Present Value of Payments or Deposits That Grow at a Constant Rate If the First Payment Is Due One Year From Now

Keystrokes	***Explanation***
, C ALL	clearing memory
, BEG/END	only if BEGIN displayed
30000, ÷, 1.05, =, PMT	28,571.43 displayed as the adjusted first payment
1.08, ÷, 1.05, -, 1, ×, 100, =, I/YR	adjusts the discount rate (8%) by the inflation rate (5%) for an adjusted rate of 2.86
10, N	number of payments
PV	-245,506.62 displayed

Fisher Effect

This concept, called the *Fisher Effect*, can be expressed in equation form as follows:

$$\text{nominal rate of return} = [(1 + \text{real rate}) \times (1 + \text{inflation rate})] - 1$$

where all rates are expressed as decimals.

Inserting the data from the example:

$$\text{nominal rate of return} = (1.08 \times 1.05) - 1 = 0.134$$

That is, 13.4 percent is the nominal rate required to earn a real rate of 8 percent if the inflation rate is 5 percent.

In order to solve these problems as an annuity, you have to modify the interest rate to reflect both the investment returns and the growth or inflation rate. Every cash flow has to be subject to both rates in order for the modified interest rate to solve the problem. If the first cash flow is not subject to the growth/inflation rate, then you need to adjust it by that rate to solve the problem. That is what was done in the text box above. When the first cash flow takes place today (annuity due), then no adjustment is necessary. There, instead of adjusting the first cash flow, the present value is adjusted to reflect the timing of cash flows. In other words, the problem is solved as an annuity due and then is converted to an ordinary annuity.

Future Value of Uneven Cash Flows

Compounding a series of uneven cash flows to a future value utilizes our usual approach to calculating FVSS problems.

For example, assume that a business plans to make a deposit each year to fund certain pension obligations as follows:

End of Year	Amount of Deposit
1	$60,000
2	40,000
3–5	50,000
6–10	60,000

If these deposits earn 10 percent compound annual interest, how much will be in the account after 10 years? In other words, what is the future value of this series of uneven cash flows?

The solution to this problem requires compounding each individual deposit from the time it is made until the end of the 10th year, then totaling the individual FVSS to obtain the answer. Thus, the first $60,000 deposit should be compounded for 9 years at 10 percent to produce its FVSS. The second deposit should be compounded for 8 years, the third for 7 years, and so on. The final $60,000 deposit, of course, earns no interest because it is made at the end of the 10th year.

Using the FVSS and the FVA Formulas to Compute the Future Value of Uneven Cash Flows

Continuing with the pension funding example, the FVSS and the FVA formulas can be used to compute the future value of the series of pension funding deposits (a series of uneven cash flows) earning 10 percent compound annual interest.

The FVSS formula is

$$\text{FVSS} = \text{PVSS} \times (1 + i)^n$$

where $(1 + i)^n$ is the FVSS factor. Because $(1 + i)^n$ is the FVSS factor, the FVSS formula can be written as

$$\text{FVSS} = \text{PVSS} \times \text{FVSS factor}$$

The FVA formula is

$$\text{FVA} = \text{annual deposit} \times \{[(1 + i)^n - 1] \div i\}$$

where the bracketed portion of the equation is the FVA factor. Because the bracketed portion is the FVA factor, the FVA formula can be written as

$$\text{FVA} = \text{annual deposit} \times \text{FVA factor}$$

Using the FVSS and the FVA formulas with the appropriate factor tables (located in appendix F), the solution to the pension funding problem requires the following six steps:

In step 1, calculate the FVSS for the $60,000 deposit at the end of year 1.

$$\begin{aligned} \text{FVSS} &= \text{PVSS} \times (1 + .10)^9 \\ &= \text{PVSS} \times \text{FVSS factor} \\ &= \$60{,}000 \times 2.3579 \\ &= \$141{,}474.00 \end{aligned}$$

In step 2, calculate the FVSS for the $40,000 deposit at the end of year 2.

$$\begin{aligned} \text{FVSS} &= \text{PVSS} \times (1 + .10)^8 \\ &= \text{PVSS} \times \text{FVSS factor} \\ &= \$40{,}000 \times 2.1436 \\ &= \$85{,}744.00 \end{aligned}$$

In step 3, calculate the FVA of 3 years for the $50,000 deposits at the end of years 3 through 5.

$$\begin{aligned} \text{FVA} &= \text{annual deposit} \times \{[(1 + 10)^3 - 1] \div .10\} \\ &= \text{annual deposit} \times \text{FVA factor} \\ &= \$50{,}000 \times 3.3100 \\ &= \$165{,}500.00 \end{aligned}$$

In step 4, calculate the FVSS for the FVA (determined in step 3) of $165,500.00.

$$\begin{aligned} \text{FVSS} &= \text{PVSS} \times (1 + .10)^5 \\ &= \text{PVSS} \times \text{FVSS factor} \\ &= \$165{,}500 \times 1.6105 \\ &= \$266{,}537.75 \end{aligned}$$

In step 5, calculate the FVA of 5 years for the $60,000 deposits at the end of years 6 through 10.

$$\begin{aligned} \text{FVA} &= \text{annual deposit} \times [(1 + 10)^5 - 1] \div .10 \\ &= \text{annual deposit} \times \text{FVA factor} \\ &= \$60{,}000 \times 6.1051 \\ &= \$366{,}306.00 \end{aligned}$$

In step 6, total the future values of the deposits from steps 1, 2, 4, and 5.

$141,474.00	(FVSS from step 1)
85,744.00	(FVSS from step 2)
266,537.75	(FVSS from step 4)
366,306.00	(FVA from step 5)
$860,061.75	

Using an HP-10BII to Solve Future Value Problems With Uneven Cash Flows

The HP-10BII is unable to directly compute the future value of a series of uneven cash flows. Therefore, the way to solve this type of problem with an HP-10BII is to divide it into two separate parts. The first part requires you to compute the present value of a series of uneven cash flows; the second part requires you to compute the future value of the present value determined in part one.

For example, assume the following deposits to a savings account:

End of Year	Amount of Deposit
1	$500
2–5	600
6–8	700
9	200

If these deposits earn 7 percent compound annual interest, how much will be in the account after 9 years?

To solve the first part of the problem you have to compute the present value of the deposits. First clear your HP-10BII's memory. (Remember, as previously noted, the CF_j key assumes that deposits are made immediately. Therefore, it makes no difference whether your HP-10BII is set for beginning-of-year or end-of-year deposits. Either way the answer will be the same.) Then enter 0 and CF_j (because there is no deposit at the start of year one); 500 and CF_j (for the first deposit at the end of year 1); 600, CF_j, 4, , and N_j (for the next four deposits for years 2 through 5); 700, CF_j, 3, , and N_j (for the next three deposits for years 6 through 8); 200 and CF_j (for the final deposit for year 9). Next enter 7 and I/YR (for the interest rate). Finally enter and NPV to find the present value of this series of uneven cash flows or deposits, which is $3,785.22.

To solve the second part of the problem you need to compute the future value of the present value determined in part one. This requires you to carry the present value from part one forward as a single sum to the end of year 9 using the same 7 percent compound annual interest rate. Thus, enter +/-, PV; 9 and

N. Then enter FV to find the future value of this set of uneven cash flows or deposits, which is $6,958.97.

HP-10BII: Keystrokes for Computing the Future Value of a Series of Uneven (End-of-Year) Cash Flows	
Part 1 Keystrokes	***Explanation***
, C ALL	clearing memory
0, CF_j	no deposit at beginning of first year
500, CF_j	deposit at end of first year
600, CF_j, 4, , N_j	4 consecutive deposits for years 2 through 5
700, CF_j, 3, , N_j	3 consecutive deposits for years 6 through 8
200, CF_j	final deposit for year 9
7, I/YR	discount interest year
, NPV	3,785.22 displayed as the present value
Part 2 Keystrokes	***Explanation***
+/-, PV	-3,785.22 displayed as the present value from Part 1
9, N	number of payments
FV	6,958.97 displayed as future value

Payments or Deposits Growing by a Constant Percentage. Now we will move from uneven irregular cash flows to those that increase each year by a constant percentage. This type of problem frequently arises when a client sets up a savings plan and the annual contribution to the plan increases each year at some assumed rate.

For example, assume that you plan to begin a program of annual saving, beginning with a $5,000 deposit now. Assume also that you expect your income and therefore the amount you plan to save to increase at an annual rate of 10 percent. If your savings earn a 7 percent compound annual rate of interest, how much will be in your account after 5 years? What is the future value, compounded at 7 percent interest, of this 5-year stream of deposits that increase at an annual rate of 10 percent?

HP-10BII: Keystrokes for Computing the Future Value of a Series of Beginning-of-Year Deposits That Increase at a Constant Rate

Part 1 Keystrokes	***Explanation* (Finding the PV)**
, C ALL	clearing memory
, BEG/END	only if BEGIN not displayed
5000, PMT	first deposit
1.07, ÷, 1.10, -, 1, x, 100, =, I/YR	adjusts the discount rate (7%) by the growth rate (10%) for an adjusted rate of -2.73
5, N	number of payments
PV	-26,441.73 displayed as the present value of the deposits
Part 2 Keystrokes	***Explanation (Finding the FV of the PV in Part I)***
PV	-26,441.73 displayed as the present value from Part 1
7, I/YR	restores original interest rate
0, PMT	no additional deposits
FV	37,085.89 displayed as the future value of the deposits
Without Reentering All the Information If the Deposits Occur at the End of Each Year	
FV	37,085.89 displayed
÷, 1.07, =	34,659.71 displayed as the future value of the deposits

One approach to solving this type of problem is to use a formula first that takes growth into consideration. However, rather than attempting to solve this problem by using a formula, it is much easier and quicker to use your HP-10BII. The procedure is similar to that used in calculating the future value of a series of uneven cash flows, which requires that the problem be divided into two parts. In part one of the problem, compute the present value of the series of increasing cash flows. In part two compute the future value of the present value determined in part one.

Begin by setting your HP-10BII for beginning-of-year payments or deposits. (Since the problem does not involve the CF_j key, it is necessary to set your HP-10BII to reflect the proper payment mode.) Next clear its memory. Then for part one of the problem enter 5000 and PMT (for the first deposit); 1.07, ÷, 1.10, -, 1, x, 100, =, and I/YR (for the inflation-adjusted interest rate of -2.73); 5 and N (for the number of payments). Then solve part one by entering PV (for the present value of the series of increasing deposits), and $26,441.73 will appear on the screen.

For part two enter 7 and I/YR (to restore the original compounding rate); 0 and PMT (to clear out extraneous data). Then to solve the problem and find the future value of the series of beginning-of-year increasing deposits enter FV and the answer, $37,085.89, will be displayed on the screen. However, before clearing your HP-10BII's memory, the future value of end-of-year increasing deposits can be found by first entering FV and then entering ÷, 1.07, and =. The answer displayed on the screen will be $34,659.71.

EVALUATING INVESTMENTS WITH DISCOUNTED CASH FLOW ANALYSIS

One of the most common personal and business applications of time-value-of-money principles and techniques is the evaluation of a proposed investment. For example, assume that you are considering the purchase of a bond that involves a cash outflow now (the purchase price) and a series of cash inflows over several time periods in the future (the interest payments and the face amount). Or perhaps you are considering construction of an apartment building that involves a cash outflow now and perhaps again next year (the construction costs), after which you anticipate a series of cash inflows for a period of years (the rental payments). Or perhaps the investment under consideration is the purchase (cash outflow) of a piece of equipment that reduces expenses (cash inflow) over some future period.

In all of these situations there is a trade-off: one or more cash outflows in return for one or more cash inflows. Are the inflows expected worth the outflows expended? Is the investment a good one? These questions cannot be answered adequately without taking the time value of money into account. Discounted cash flow analysis can be used to assist you in evaluating an investment by making the time value of cash outflows and inflows comparable. It can help you decide (1) whether a proposed investment opportunity is an acceptable one and (2) how to rank several competing investment opportunities in terms of their relative acceptability.

There is more, however, to evaluating an investment opportunity than simply crunching some numbers through various time-value-of-money formulas or calculator functions. The degree of certainty in amount, timing, and duration associated with various cash outflows and inflows, as well as the tax aspects, must also be considered. Evaluation of tax aspects is beyond the scope of this chapter. Risk considerations, however, are incorporated in the interest rate that is used in discounted cash flow analysis.

As you will see, the mechanics of discounted cash flow analysis are quite straightforward when the pattern of the cash outflows and inflows is simple. Calculations become more complex when the amounts vary from year to year, and still more complex when the amounts are positive in some years and negative in others. The remainder of this chapter deals with discounted cash flow analysis situations, first showing simple applications then gradually demonstrating more complex ones.

Discounted Cash Flow Techniques Defined

There are two commonly used techniques for discounted cash flow analysis: (1) calculation of an investment's net present value (NPV) and (2) calculation of its internal rate of return (IRR).

Net Present Value

net present value (NPV)

The *net present value (NPV)* of an investment is the present value of the stream of cash inflows minus the present value of the stream of cash outflows. The interest rate selected by the investor for discounting the two cash flow streams is either the minimum rate of return acceptable to the investor in light of his or her assessment of the investment's risk, the investor's cost of capital to fund the investment, or the rate available on alternative investment opportunities with a similar degree of risk.

If the result of the NPV calculation is positive, that is, if the present value of the cash inflow stream exceeds the present value of the cash outflow stream, the investment is a good one and the investor's wealth will increase (in a time value sense) by the amount of the positive NPV. A positive NPV means that the rate of return provided by the investment (whatever that rate is) exceeds the interest rate selected by the investor as the discount rate. If the NPV is negative, the reverse is true and the investor's wealth will decrease (in a time value sense) by the amount of the negative NPV. And if the NPV of the investment is zero, the investor's wealth will neither increase nor decrease (in a time value sense), leaving him or her indifferent as to whether the investment should be made.

Internal Rate of Return

internal rate of return (IRR)

The *internal rate of return (IRR)* on an investment is the discount rate at which the present value of the stream of cash inflows (investment returns) equals the present value of the stream of cash outflows (investment costs). This is to say that the IRR on an investment is the discount rate at which the NPV of the investment is zero. If the investment's IRR is larger than the minimum rate acceptable by the investor, the investment is a good one and should be undertaken. If not, it should be rejected. The investor's choices for determining an acceptable minimum rate of return are the same as those described in selecting a discount rate for a net present value calculation.

Similarity of the NPV and IRR Techniques. Note that the NPV and IRR methods of evaluating an investment are very similar. In computing NPV, the investor specifies the minimum acceptable interest rate and determines whether at that rate the present value of the inflows exceeds the present value of the outflows. In computing IRR, the investor ascertains the interest rate that makes the present value of the inflows equal to the present value of the outflows; that is, the investor ascertains the interest rate that produces an NPV of zero.

As a general rule, when used to evaluate a particular investment, the two techniques will lead the investor to the same conclusion. If the NPV is positive, the IRR normally is acceptable and the investment is an attractive one. Conversely, if the NPV is negative, the IRR normally is unacceptable and the investment should be rejected. When used to rank several investments as to their relative acceptability, however, there are situations in which the NPV and IRR techniques produce different results using the same data. In most cases, the NPV method is more reliable for ranking several competing investment possibilities. The project with the largest NPV should be ranked first, the one with the next largest NPV should be second, and so on.

Solving Simple NPV Problems With Formulas

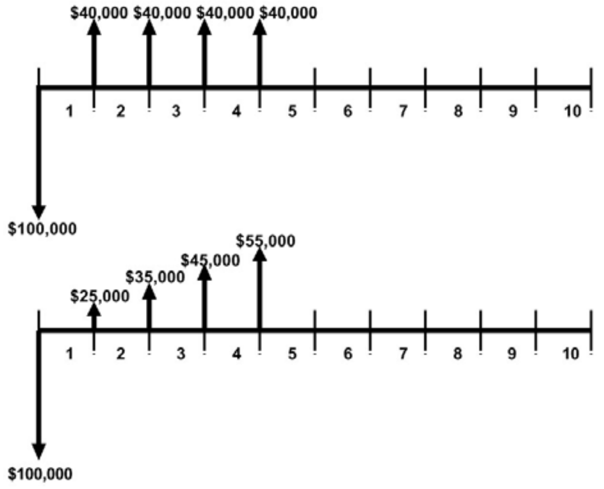
Assume that you are evaluating a project that requires purchasing a piece of equipment that costs $100,000. The project is expected to provide a positive net cash flow of $40,000 per year at the end of each of the next 4 years, after which it will have no value. If your cost of capital is 10 percent per year, should you invest in this project?

The cash flows associated with this project are illustrated in the time line in the upper half of *Figure 7-3, Time Line Depiction of NPV Problem: Level and Uneven Inflows*. The sole cash outflow, $100,000, occurs immediately, so its present value is $100,000. The four cash inflows of $40,000, totaling $160,000, are a regular annuity for 4 years, and should be discounted at the 10 percent cost of funds. Using the PVA formula with the appropriate PVA factor, you can find the present value of the cash inflows as follows:

$$\text{PVA} = \text{annual payment} \times \{1 - [1 \div (1 + i)^n] \div i\}$$

The bracketed portion of the equation is the PVA factor.

Figure 7–3 Time Line Depiction of NPV Problem: Level and Uneven Inflows



The upper time line depicts an investment that entails an initial cash outflow of $100,000 and a level stream of four $40,000 cash inflows. In the lower time line, the initial cash outflow is also $100,000, but the four cash inflows increase over time.

$$\begin{aligned} \text{PVA} &= \$40{,}000 \times \{1 - [1 \div (1.10)^4] \div .10\} \\ &= \$40{,}000 \times 3.1699 \\ &= \$126{,}796.00 \end{aligned}$$

Subtracting the present value of the cash outflow from the present value of the stream of cash inflows ($126,796.00 - $100,000.00 = $26,796.00) results in a positive NPV of $26,796.00. This positive NPV indicates that your wealth will increase by $26,796 (in today's dollars) if you invest in the project.

Even when a problem involves uneven cash flows, the approach to solving for NPV remains the same. For example, assume that another project with the same $100,000 initial cost will create increasing cash inflows that total $160,000, as shown on the bottom half of *Figure 7-3, Time Line Depiction of NPV Problem: Level and Uneven Inflows*. In this situation, you can calculate the PVSS for each cash inflow and then total the four PVSS amounts to find the present value of the cash inflows. Thus, using the PVSS formula with the appropriate PVSS factor, you can calculate the PVSS for each of the cash inflows as follows:

$$\text{PVSS} = \text{FVSS} \times [1 \div (1 + i)^n]$$

The bracketed portion of the equation is the PVSS factor.

1st PVSS = \$25,000 × [1 ÷ $(1.10)^1$] = \$25,000 × .9091 = \$22,727.50

2d PVSS = \$35,000 × [1 ÷ $(1.10)^2$] = \$35,000 × .8264 = \$28,924.00

3d PVSS = \$45,000 × [1 ÷ $(1.10)^3$] = \$45,000 × .7513 = \$33,808.50

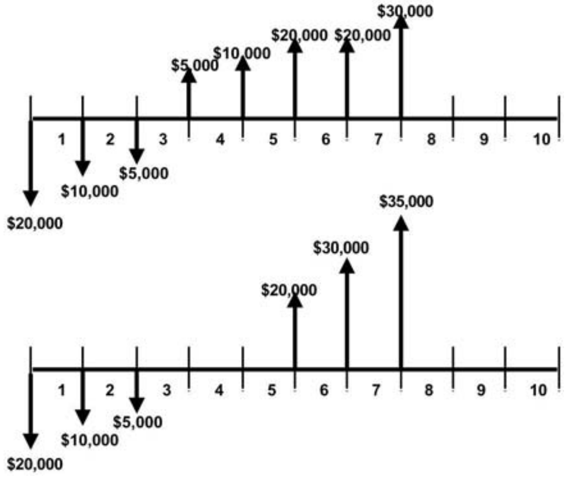
4th PVSS = \$55,000 × [1 ÷ $(1.10)^4$] = \$55,000 × .6830 = <u>\$37,565.00</u>

Sum of the PVSS inflows from each year = \$123,025.00

The sum of the four PVSS is \$123,025.00, which is the present value of the stream of increasing cash inflows. Subtracting the present value of the cash outflow from the present value of the stream of cash inflows (\$123,025.00 - \$100,000.00 = \$23,025.00) results in a positive NPV of \$23,025.00. Notice that although both projects generate the same total undiscounted cash inflow of \$160,000, the project with the increasing pattern of cash inflows has a lower NPV because its higher cash inflows occur further in the future.

The entire costs of the projects in the two problems depicted in *Figure 7-3, Time Line Depiction of NPV Problem: Level and Uneven Inflows* were incurred at their inceptions. Cash outflows, however, may occur at any time during the lifetime of a project. For example, the top half of *Figure 7-4, Time Line Depiction of NPV Problems: A Series of Cash Outflows and Cash Inflows and Years of Zero Outflows or Inflows* depicts a project in which there are 3 years of uneven cash outflows (\$20,000, \$10,000, and \$5,000) followed by 5 years of uneven cash inflows (\$5,000, \$10,000, \$20,000, \$20,000, and \$30,000). As in the earlier calculations, to compute the NPV for this project, you need to subtract the present value of the series of cash outflows from the present value of the series of cash inflows. Note also from the *Figure 7-4, Time Line Depiction of NPV Problems: A Series of Cash Outflows and Cash Inflows and Years of Zero Outflows or Inflows* upper time line that the cash outflows occur at the beginning of years 1, 2, and 3, while the cash inflows occur at the end of years 3, 4, 5, 6, and 7.

Figure 7–4 Time Line Depiction of NPV Problems: A Series of Cash Outflows and Cash Inflows and Years of Zero Outflows or Inflows



The upper time line depicts a project with a decreasing series of cash outflows followed by an increasing series of cash inflows. The lower time line depicts a similar project with a 3-year gap between the last cash outflow and the first cash inflow. Already you should expect that the NPV of the upper time line project would exceed that of the lower time line project because the same dollar amount of undiscounted cash inflows occurs earlier in the upper project.

If you determine that the minimum acceptable rate of return is 8 percent, you can use the PVSS formula with the appropriate PVSS factor to calculate the PVSS for each of the cash outflows and inflows. Thus the PVSS for each cash outflow is

1st PVSS = \$20,000 × [1 ÷ $(1.08)^0$] = \$20,000 × 1.0000 =	\$20,000.00
2d PVSS = \$10,000 × [1 ÷ $(1.08)^1$] = \$10,000 × .9259 =	\$9,259.00
3d PVSS = \$5,000 × [1 ÷ $(1.08)^2$] = \$5,000 × .7513 =	\$4,286.50
Sum of the PVSS outflows from each year =	\$33,545.50

The sum of these three PVSS is $33,545.50, which is the present value of the stream of decreasing cash outflows. The PVSS for each cash inflow is

1st PVSS = \$5,000 × [1 ÷ $(1.08)^3$] = \$5,000 × .7938 = \$3,969.00
2d PVSS = \$10,000 × [1 ÷ $(1.08)^4$] = \$10,000 × .7350 = \$7,350.00
3d PVSS = \$20,000 × [1 ÷ $(1.08)^5$] = \$20,000 × .6806 = \$13,612.00
4th PVSS = \$20,000 × [1 ÷ $(1.08)^6$] = \$20,000 × .6302 = \$12,604.00
5th PVSS = \$30,000 × [1 ÷ $(1.08)^7$] = \$30,000 × .5835 = \$17,505.00
Sum of the PVSS inflows from each year = \$55,040.00

The sum of these five PVSS is \$55,040.00, which is the present value of the stream of increasing cash inflows. Subtracting the present value of the stream of cash outflows from the present value of the stream of cash inflows (\$55,040.00 - \$33,545.50 = \$21,494.50) results in a positive NPV of \$21,494.50.

This approach isolated the cash inflows from the cash outflows. However, isolating the cash flows by direction (in or out) is not necessary as long as you remember to use a negative sign for each outflow. Then, when you sum the PVSS amounts, you will be combining both positive (adding) and negative (subtracting) numbers.

There may be one or more years in which there is neither a cash outflow nor a cash inflow. The procedure for solving these problems remains the same. A time line will be helpful for confirming that inflows and outflows are discounted for the correct number of years. For example, if the previous problem (depicted on the top time line of *Figure 7-4, Time Line Depiction of NPV Problems: A Series of Cash Outflows and Cash Inflows and Years of Zero Outflows or Inflows*) is changed so that there is no cash inflow at the end of years 3 and 4 even though there is still the same dollar amount of undiscounted cash inflows in the problem (compare the bottom time line in *Figure 7-4, Time Line Depiction of NPV Problems: A Series of Cash Outflows and Cash Inflows and Years of Zero Outflows or Inflows* with the top time line), then the present value of the stream of cash flows (that is, both inflows and outflows) becomes:

1st PVSS = – \$20,000 × [1 ÷ $(1.08)^0$] = – \$20,000 × 1.0000 = – \$20,000
2d PVSS = – \$10,000 × [1 ÷ $(1.08)^1$] = – \$10,000 × .9259 = – \$9,259.00
3d PVSS = – \$5,000 × [1 ÷ $(1.08)^2$] = – \$5,000 × .8573 = – \$4,286.50
4th PVSS = \$20,000 × [1 ÷ $(1.08)^5$] = \$20,000 × .6806 = \$13,612.00
5th PVSS = \$30,000 × [1 ÷ $(1.08)^6$] = \$30,000 × .6302 = \$18,906.00
6th PVSS = \$35,000 × [1 ÷ $(1.08)^7$] = \$35,000 × .5835 = \$20,422.50
Sum of the PVSS flows from each year = \$19,395.00

Thus, combining both the PVSS cash outflows (negative amounts) and the PVSS cash inflows (positive amounts) results in a positive NPV of \$19,395.00. As suggested in *Figure 7-4, Time Line Depiction of NPV Problems: A Series of Cash Outflows and Cash Inflows and Years of Zero Outflows or Inflows*, you would expect a lower NPV in the bottom time line (\$19,395.00) compared with the top time line ($21,494.50) because even though the dollar amount of the undiscounted cash inflows is the same in both time lines, the cash inflows occur earlier in the upper time line.

Finally, in some situations there may be both cash inflows and cash outflows in the same year. In these cases, first subtract the year's outflows from its inflows. Next, determine the PVSS of the year's net cash flows, whether they be positive (inflows) or negative (outflows). Then, as before, sum the PVSS from each year with a net cash flow whether that flow is positive or negative. The result is the NPV. If positive, the project should be undertaken because it will increase your wealth (in today's dollars). If negative, the project is a bad investment and should be rejected.

As net present value problems become more involved, formula solutions become cumbersome and time consuming. At this juncture, it is necessary to pick up your HP-10BII.

Solving Complex NPV Problems With an HP-10BII

Ungrouped Cash Flows

Now we will take up somewhat more complicated problems, starting with NPV problems in which there are no consecutive cash flows of the same amount and sign. To solve this type of problem with your HP-10BII, you must use the CF_j (cash flow) key (explained earlier in this chapter). In addition, you must also use the +/- key to enter outflows as negative numbers. Finally, you need to enter the discount rate, as before, and use the NPV key to produce the answer.

To illustrate the process, assume you have been asked to make a $75,000 loan. The borrower agrees to the following repayment schedule

End of Year	Amount of Payment
1	$ 0
2	15,000
3	20,000
4	25,000
5	30,000

Should you make the loan if you insist on a rate of return of at least 11 percent on your investment?

Using your HP-10BII, first clear its memory. Then enter 75000, +/-, and CF_j. The loan (an outflow at the start of the first year) has now been entered as a negative amount. Next enter 0 and CF_j (because there is no net cash inflow or outflow at the end of the first year). Then enter (the remaining cash inflows) 15000 and CF_j; 20000 and CFj; 25000 and CF_j; 30000 and CF_j. Now enter 11 and I/YR (the minimum acceptable rate of return). Finally, press and NPV to produce the answer of –$13,930.02.

Obviously, the loan has a negative NPV and should not be made since, in light of the time value of money, it would cost you almost $14,000. This loan would be unacceptable even if you were willing to settle for a 5 percent rate of return. To check this (without clearing your HP-10BII's memory and starting over) enter 5 and I/YR. Then enter and NPV and you will display a negative NPV of –$44.46.

HP-10BII: Keystrokes for Computing the Net Present Value (NPV)	
Keystrokes	***Explanation***
, C ALL	clearing memory
75000, +/-, CF_j	to enter the amount of the loan as an outflow
0, CF_j	no cash flow at the end of year 1
15000, CF_j	end of year 2 cash flow
20000, CF_j	end of year 3 cash flow
25000, CF_j	end of year 4 cash flow
30000, CF_j	end of year 5 cash flow
11, I/YR	the required rate of return
, NPV	–13,930.02 displayed as the NPV

Grouped Cash Flows

Your HP-10BII can solve NPV problems involving more cash flows if there is grouped data. As explained earlier in this chapter, for such problems you should enter, along with the amount of each cash flow, the total number of times it occurs in succession including the first time.

To illustrate the process, assume you plan to invest in an oil exploration limited partnership that entails the following cash flows:

Timing of Flow	Amount of Cash Flow
Immediately	$50,000 outflow
Years 1 through 5	0
End of years 6 through 9	6,000 inflow
End of year 10	60,000 inflow

Is this investment acceptable if you insist on a rate of return of at least 12 percent per year?

First clear your HP-10BII's memory and then enter 50000, +/-, and CF_j (for the initial cash outflow). Next enter 0, CF_j, 5, , and N_j (for the years 1 through 5 with no cash flows). Then enter 6000, CF_j, 4, , and N_j (for the inflows at the end of years 6 through 9). Then enter 60000 and CF_j (for the last inflow at the end of year 10). Next enter 12 and I/YR (for the required interest rate). Finally enter and NPV to compute the answer, which is –$20,340.76.

HP-10BII: Keystrokes for Computing the NPV of a Project With Grouped Cash Flows	
Keystrokes	***Explanation***
, C ALL	clearing memory
50000, +/-, CF_j	the investment as a cash outflow
0, CF_j, 5, , N_j	no cash flows for years 1 through 5
6000, CF_j, 4, , N_j	4 consecutive cash inflows for years 6 through 9
60000, CF_j	the final cash inflow for year 10
12, I/YR	the required rate of return
, NPV	-20,340.76 displayed as the NPV

Solving IRR Problems With an HP-10BII

Recall that the internal rate of return is the discount rate that equates the present value of cash inflows and outflows, producing an NPV of zero. If the IRR exceeds the investor's minimum desired rate of return, the investment is an attractive one. If it equals the minimum desired rate, the investor should be neutral toward the project. If the IRR is below the minimum desired rate, the investment is unattractive.

The computation of the IRR of an investment is an extremely complex process that can be conducted manually using a trial and error method. Beginning with an estimate of the IRR, such as 10 or 15 percent, discount the cash inflows and outflows to their present values. If the present value of the inflows exceeds the present value of the outflows, so that NPV is positive at the selected rate, choose a higher rate and repeat the calculations. You are choosing a higher rate because you need a higher rate to reduce the NPV to (or close to) zero. On the other hand, if the discounted cash outflows exceed the discounted cash inflows resulting in a negative NPV, choose a lower interest rate and repeat the calculations. Continue this trial-and-error process, adjusting the discount rate until the resulting NPV is equal to (or close to) zero. The discount rate that produces an NPV of zero is the IRR, an average rate of return over the time horizon weighted to reflect the amount and timing of the various cash flows.

When using your HP-10BII to solve for the IRR, do not enter a discount (interest) rate because it is the unknown value that you are solving for. Begin by clearing the HP-10BII's memory and then proceed by entering all the cash flows. After the cash flows have been entered, press and IRR/YR for the answer. The IRR/YR key is to the left of the NPV key.

For example, to solve a simple IRR problem involving a single cash outflow of $10,000 followed by five cash inflows of $2,300 each, first set your HP-10BII to display four decimal places by entering , DISP and 4. (Remember 4 decimal precision is desired in some interest rate calculations.)[45] Then clear your HP-10BII's memory and enter 10000, +/- and CF_j (for the cash outflow). Next enter 2300, CF_j, 5, , and N_j (for the five cash inflows). Finally enter and IRR/YR to compute the answer, which is 4.8472.

HP-10BII: Keystrokes for Solving a Simple IRR Problem	
Keystrokes	***Explanation***
, C ALL	clearing memory
, DISP, 4	set at 4 decimal precision
10000, +/-, CF_j	the initial cash outflow
2300, CF_j, 5, , N_j	5 consecutive cash inflows
, IRR/YR	4.8472 displayed as the IRR

To take a more complex IRR example, assume the investment you are considering involves an immediate $6,000 cash outflow followed by a series of cash flows.

Timing of Flow	**Amount of Cash Flow**
Immediately	$6,000 outflow
End of year 1	200 inflow
End of years 2 through 5	900 inflow
End of year 6	1,000 outflow
End of year 7	1,100 inflow
Year 8	0
End of year 9	1,200 inflow
End of year 10	3,000 inflow

To solve for the IRR of this investment, first make sure that your HP-10BII is set for 4 decimal precision (if you have not already done so) and then clear its memory. Then enter 6000, +/-, and CF_j (for the immediate cash outflow). Next

45. Four decimal precision is useful in time-value-of-money problems because the calculator will round to the nearest basis point. A basis point is 1/100 of a percentage point. A basis point, expressed as a decimal, is .0001.

enter 200 and CF_j (for the cash inflow at the end of year 1); 900, CF_j, 4, , and N_j (for the cash inflows at the end of years 2 through 5); 1000, +/-, and CF_j (for the cash outflow at the end of year 6); 1100 and CF_j (for the cash inflow at the end of year 7); 0 and CF_j (because there are no cash flows in year 8); 1200 and CF_j (for the cash inflow at the end of year 9); 3000 and CF_j (for the cash inflow at the end of year 10). Finally enter and IRR/YR to compute the IRR, which is 4.6664. After solving for the IRR and without reentering the cash flows, you can choose a discount (interest) rate, such as 10 percent (enter 10 and I/YR), and solve for the NPV (enter and NPV).

HP-10BII: Keystrokes for Solving a Complex IRR Problem	
Keystrokes	***Explanation***
, C ALL	***clearing memory***
6000, +/-, CF_j	immediate cash outflow
200, CF_j	end of year 1 cash inflow
900, CF_j, 4, , N_j	end of years 2 through 5 cash inflow
1000, +/-, CF_j	end of year 6 cash outflow
1100, CF_j	end of year 7 cash inflow
0, CF_j	no cash flows in year 8
1200, CF_j	end of year 9 cash inflow
3000, CF_j	end of year 10 cash inflow
, IRR/YR	4.6664 displayed as the IRR
Keystrokes for Computing the NPV Without Reentering All the Information	
10, I/YR	discount interest rate
, NPV	-1,559.1085 displayed as the NPV

Problems in Using IRR as a Decision Rule

As previously indicated, the IRR provided by potential investments must be used with caution in decision making. This is especially true when the IRR is used as a means for ranking competing investment opportunities.

One situation in which an evaluation based solely on comparative IRRs may lead to an incorrect decision is the case of mutually exclusive investment projects that are of substantially different magnitudes. For example, assume you can invest in either of two pieces of equipment for your business, but the investment in one eliminates the possibility of investing in the other. (These are called mutually exclusive investments.) Also assume you can borrow enough money to make the investment in either piece of equipment, neither of which has any salvage value. The total cash flows from the two projects, A and B, are shown below:

Beginning of Year	Cash Flows (A)	Cash Flows (B)
1	$5,000 outflow	$50,000 outflow
2	3,000 inflow	25,000 inflow
3	4,000 inflow	35,000 inflow
4	5,000 inflow	45,000 inflow

The IRR from project A is 54.0603 percent, while the IRR from project B is only 42.9811 percent. (You should confirm these results on your own HP-10BII.) Does this mean that project A is preferable? Not necessarily. For example, if the money to meet the initial outflow is borrowed at a 10 percent rate of interest, it may be preferable to invest in project B. On the basis of a 10 percent discount rate, the NPV of project B is $35,462.06, whereas the NPV of project A is only $4,789.63. In other words, project B increases your wealth (in a time-value sense) by more than seven times the amount of project A. A more modest rate of return on a large project may be preferable to a higher rate of return on a small project. (Note, however, that this illustration deals with mutually exclusive projects. If the projects are not mutually exclusive, then the correct investment decision might be to invest in both projects.)

There are several other situations in which a decision based on the IRR may be incorrect. As in the case of mutually exclusive investment projects with substantially different magnitudes, mutually exclusive investment projects with substantially different cash flow patterns may also lead to an incorrect decision. There is also difficulty in relying solely on IRRs when deciding between investment opportunities that have substantially different durations. Another limitation of the IRR method for comparing investment projects involves the question of financing costs of cash outflows that may be associated with one or both of the projects. A final problem concerning the IRR method is that it is possible for a project to have multiple IRRs, each of which is mathematically correct. It is also possible that an investment may have no IRR within the realm of real numbers. What should you do about these problems of using the IRR as a means of ranking competing investment opportunities? One simple solution is to forget about using the IRR as a method for deciding between investment opportunities and use the NPV method instead.

INCREASING THE COMPOUNDING, DISCOUNTING, OR PAYMENT FREQUENCY

All the explanations and illustrations so far in this and the preceding chapter are based on the assumption that compounding and discounting occur once per year. In reality, however, compounding and discounting often occur more frequently than annually. For example, a certificate of deposit can be credited with compound interest on a monthly basis. A traditional savings account can earn daily compound interest. The present value of the income from a corporate bond typically is computed on a semiannual discounting basis.

In addition, all of the explanations of problems involving periodic payments thus far have been based on the assumption that payments are made annually. Often, however, such payments occur more than once per year. Installment loan payments, for example, are frequently made monthly. Bond interest payments are usually made every 6 months. Deposits into the savings accounts of many people are made weekly.

The remainder of this chapter examines the effect on the interest rate and, therefore, on the time value of money resulting from compounding or discounting more often than annually. It also contains an explanation of how to solve problems involving a series of level payments that occur more often than once per year regardless of how often compounding or discounting takes place.

Nominal Versus Effective Interest Rates

conversion of interest earnings into principal

As you know, compounding results in the *conversion of interest earnings into principal.* For example, if $100 is deposited in an account today and the account is subsequently credited with $7.00 of compound interest, the principal on which future interest is credited rises to $107. The $7.00 of interest, when paid, converts into and becomes principal.

If compounding occurs annually, when does interest convert into principal and begin to earn interest on itself? Obviously, the conversion and thus the capacity for increased interest earnings occur after one year, again after 2 years, and so on. If, on the other hand, compounding occurs on a monthly basis, when does interest convert into principal and thus begin to earn interest on itself? The conversions occur after one month has elapsed, again after 2 months, 3 months, and so on.

Of course, the greater the frequency with which compounding occurs, the smaller the dollar amount of interest earned and converted into principal on the occasion of each compounding. Naturally, the amount of interest a given amount of principal can earn in a week is less than it can earn in a month at any given stated or nominal annual interest rate. Nevertheless, all other things being equal, the more frequent the compounding, the greater the total interest credited to an account for the year. To illustrate, *Table 7-3, Total Interest Credited to a $10,000 Deposit During One Year at a 9 Percent Stated Annual Interest Rate and Various Compounding Frequencies* shows the amount of interest credited during one year to a $10,000 deposit at a stated or nominal 9 percent annual interest rate and various compounding frequencies.

continuous compounding

Technically, of course, compounding can occur even more frequently than daily—every hour, every minute, every second, or even continuously. As the frequency increases, so does the total interest credited. The upper limit of the total interest credited to a sum of money for a particular stated or nominal annual interest rate occurs in the case of continuous compounding where interest is compounded an infinite number of times per year rather than at discrete time intervals. *Continuous compounding* is a theoretical concept useful principally in the study of advanced financial topics, but it also has some practical applications, since some financial institutions offer a product that credits interest to customer accounts on a continuous basis.

Table 7-3
Total Interest Credited to a $10,000 Deposit During One Year at a 9 Percent Stated Annual Interest Rate and Various Compounding Frequencies

Compounding Frequency	Interest Earnings
Annually	$900.00
Semiannually	920.25
Quarterly	930.83
Monthly	938.07
Weekly	940.89
Daily*	941.62

*Based on 365 days per year. Financial institutions typically use a 360-day year as the basis for daily compounding calculations. This produces a slightly smaller annual interest, all other things being equal, than if they were to use a 365-day year.

From the figures in *Table 7-3, Total Interest Credited to a $10,000 Deposit During One Year at a 9 Percent Stated Annual Interest Rate and Various Compounding Frequencies* it should be obvious that a stated or nominal annual interest rate does not necessarily reflect the true or effective interest rate. You have seen that a 9 percent nominal annual rate produces any one of six separate interest earnings in a year, depending on the frequency of compounding. Hence, it is important to distinguish between the nominal or stated annual rate (9 percent in this illustration) and the true or effective annual rate (EAR).

When compounding occurs once per year, the nominal and effective annual rates are identical. In *Table 7-3, Total Interest Credited to a $10,000 Deposit During One Year at a 9 Percent Stated Annual Interest Rate and Various Compounding Frequencies*, when annual compounding is applied, the $10,000 deposit earns $900, exactly 9 percent. In all other cases it earns more than $900 (more than 9 percent) because of the more frequent compounding.

effective annual interest rate (EAR)

The *effective annual interest rate (EAR)* is the annual rate that produces in one compounding the same amount of interest as the nominal annual rate with its compounding frequency. For instance, the 9 percent nominal annual rate in *Table 7-3, Total Interest Credited to a $10,000 Deposit During One Year at a 9 Percent Stated Annual Interest Rate and Various Compounding Frequencies*, when compounded quarterly, produces $930.83 of interest. Thus, the effective annual interest rate is 9.3083 percent ($930.83 ÷ $10,000). Similarly, the 9 percent nominal annual rate compounded daily generates $941.62 in interest. Thus, the effective annual rate is 9.4162 percent ($941.62 ÷ $10,000).

Calculating the EAR

The effective annual interest rate can be computed for any nominal rate and compounding frequency with the following formula:[46]

$$i_{eff} = [1 + (i_{nom} \div f)]^f - 1$$

where i_{eff} is the effective annual rate in decimal form,

i_{nom} is the nominal annual rate in decimal form, and

f is the compounding frequency per year

Thus, for example, a 9 percent nominal annual rate compounded monthly represents an effective annual rate of

$$i_{eff} = [1 + (.09 \div 12)]^{12} - 1$$
$$= .093807$$
$$= 9.3807\%$$

As an alternative to using this formula, you can calculate the effective annual interest rate for any nominal annual rate and compounding frequency by using your HP-10BII. To illustrate, assume that a 9 percent nominal rate is to be compounded weekly. First, set your HP-10BII to display four decimal places (if you have not already done so) and then clear its memory. Next, in order to convert a nominal rate to an effective rate, use the secondary functions on the lower half of the I/YR, PV, and PMT keys, which are NOM%, EFF%, and P/YR, respectively.

The effective annual rate is 9.4089 percent. Now before you do anything else, restore the calculator's setting to one payment period and one compounding period per year so that subsequent problems you work on will be solved correctly. This is accomplished by pressing 1, , P/YR, and C.

HP-10BII: Keystrokes for Computing the EAR	
Keystrokes	***Explanation***
, C ALL	clearing memory
, DISP, 4	to set at 4 decimal precision
9, , NOM %	to enter the nominal rate
52, , P/YR	to adjust for weekly compounding
, EFF%	to produce the effective rate of 9.4089
1, , P/YR, C	reset for one payment period per year

46. The mathematics of calculating the effective rate manually when compounding is continuous is too complex to be dealt with in this textbook.

For those who do not wish to use either a formula or your HP-10BII to determine the effective annual interest rate, appendix I will be of some help. It contains a table showing the effective annual interest rates that correspond to a number of nominal annual rates and commonly used compounding frequencies. The table also includes the effective annual rate for the unusual case of continuous compounding.

To illustrate further the difference between the nominal and effective annual rates of interest, the following list shows the effective annual rate for a nominal annual rate of 7 percent and various compounding frequencies:

7% annually = 7.0000% effective rate

7% semiannually = 7.1225% effective rate

7% quarterly = 7.1859% effective rate

7% monthly = 7.2290% effective rate

7% weekly = 7.2458% effective rate

7% daily = 7.2501% effective rate

Again, it is clear that, for a particular nominal annual interest rate, the true or effective rate rises as the frequency of compounding per year increases. Note, however, that the increase in the effective annual rate becomes smaller and smaller with each increase in compounding frequency. In the above list, for example, the change from annual to semiannual compounding increased the effective rate by .1225 percentage points (7.1225 – 7.0000). The change from semiannual to quarterly compounding changed the effective rate by only .0634 percentage points (7.1859 – 7.1225); and the change from quarterly to monthly compounding increased it by only .0432 percentage points (7.2290 – 7.1859).

Another point worth noting concerning nominal versus effective annual interest rates is that sometimes a nominal rate with a high compounding frequency produces a higher effective rate than a slightly higher nominal rate with a low compounding frequency. For example, assume that you plan to deposit $10,000 in an interest-bearing account for one year. Bank A pays interest of 8 percent compounded semiannually. Bank B pays 7.9 percent compounded daily. Where should you put your money? If you use the formula described earlier or your HP-10BII, you will find that the effective annual rate in Bank A is 8.1600 percent. Bank B, on the other hand, pays an effective annual rate of 8.2195 percent. That is an extra $5.95 credited to your $10,000 deposit if you go to Bank B.

The difference between nominal and effective annual rates inspired the Truth-in-Lending Act (TILA) and the Truth-in-Savings Act (TISA) requiring financial institutions to provide annual percentage rates (APRs) and annual percentage yields (APY). You can use an APR to calculate the EAR. Although APRs are not effective rates, they consider how frequently interest is paid.

Annual Percentage Rates

annual percentage rate (APR)

An *annual percentage rate (APR)* is the periodic rate (that is, the semiannual rate, the quarterly rate, the monthly rate, and so on) multiplied by the number of periods in a year. A daily rate is multiplied by 365, a monthly rate by 12, a quarterly rate by 4, and so on. Credit cards use a daily periodic rate that is multiplied by 365 to produce the APR for the credit card. If a credit card has a daily periodic rate of .00027123 or .027123 percent, then the card's APR is 9.8999 percent (that is, .027123 × 365) or, if rounded, 9.90 percent. It is important to recognize that the periodic rate is not the EAR divided by the number of periods in a year.

If a consumer car loan with monthly payments has an APR of 6 percent, what is its EAR?

$$EAR = (1 + APR/12)^{12} - 1$$

$$EAR = (1 + .06/12)^{12} - 1$$

$$EAR = (1 + .005)^{12} - 1$$

$$EAR = .0617 = 6.17\%$$

Note that the size of the loan, the loan term, and the payment were not needed to calculate the effective annual interest rate on the loan.

Solving for the periodic rate (PER) when you know the EAR requires some algebra skills:

$PER = (1 + EAR)^{(1/f)} - 1$, where f is the number of periods in a year;

$PER = (1.0617)^{(1/12)} - 1 = .005$;

$APR = .005 \times 12 = .06$ or 6 percent

As you can see, we are back where we started this exercise.

Impact of Compounding Frequency on Future Values

Because the effective annual interest rate rises as the frequency of compounding increases, so does the future value of a single sum. The same is true of other future values described in this and the previous chapter. For example, the future value of an annual annuity or of a series of annual uneven cash flows rises as compounding frequency increases. Conversely, in sinking fund problems the size of the annual payment needed to reach a targeted future amount diminishes as the frequency of compounding and the effective annual interest rate increase. Finally, the number of years or annual payments needed to reach a particular future value decreases as the frequency of compounding and the effective annual interest rate increase.

Impact of Discounting Frequency on Present Values

You probably have already guessed that increasing the frequency of discounting has the opposite effect on present values than increasing the frequency of compounding has on future values. This again follows from the preceding discussion of nominal versus effective interest rates.

To illustrate, calculate the present value of $100 due in one year at 7 percent. Discounted annually, you know that the effective rate is 7 percent. The present value of $100 discounted for one year at 7 percent is $93.46. Discounted quarterly, however, the effective annual rate is 7.1859 percent. The present value of $100 discounted for one year at 7.1859 percent is $93.30. Discounted monthly, the effective annual rate is 7.229 percent, which produces a present value of $93.26.

Generalizing from these results, you can conclude that, all other things being equal, an increase in the frequency of discounting increases the effective annual interest rate and therefore reduces the present value of a single sum. The same is true of the present value of an annual annuity or of a series of annual uneven cash flows. Conversely, in debt service problems, an increase in the frequency of charging interest per year, all other things being equal, increases the amount of the loan payments per year. Finally, the number of years it takes to pay off a loan or to liquidate a principal sum increases as the frequency of charging or crediting interest per year increases, all other things being equal.

Calculating the FVSS or the PVSS When Compounding or Discounting Occurs More Often Than Once per Year

When you encounter FVSS and PVSS problems in which the interest rate is compounded or discounted more frequently than annually, there are two basic ways of solving them: (1) use the effective annual interest rate or (2) adjust the nominal annual interest rate and number of periods.

Using the Effective Annual Interest Rate

The first basic approach is to solve these problems by computing the effective annual interest rate as explained earlier. Then use the effective rate in the same way you have learned to use the nominal annual interest rate throughout this and the preceding chapter.

To illustrate this approach (that is, using the effective annual interest rate) of solving this type of FVSS problem, what amount will $500 grow to in 3 years at 6 percent interest compounded quarterly? Using your HP-10BII to compute the effective rate, first clear its memory and set it for 4 decimal precision by pressing , DISP, and 4. Then enter 6, , and NOM% for the nominal rate; 4, , and P/YR for the number of compounding periods. Next enter and EFF% to find the effective annual rate. The answer, 6.1364 percent, will be displayed on the screen. Then reset your HP-10BII for one payment period per year by entering 1, , P/YR, and C.

HP-10BII: Keystrokes for Computing the EAR When Compounding Is Quarterly	
Keystrokes	***Explanation***
, C ALL	clearing memory
, DISP, 4	to set at 4 decimal precision
6, , NOM %	to enter the nominal rate
4, , P/YR	to adjust for quarterly compounding
, EFF%	6.1364 displayed as the EAR
1, , P/YR, C	reset for one payment period per year

Once having determined that the effective annual rate is 6.1364 percent (either by using your HP-10BII or appendix I), you can use it in place of the nominal 6 percent rate to solve the problem. First, remember to clear your HP-10BII's memory. Then enter 500, +/-, and PV since $500 is the present value. Next enter 6.1364 and I/YR for the effective annual rate; 3 and N for the number of periods (years). Then solve the problem by entering FV and you will have the answer, $597.81 (rounded).

To illustrate this approach (that is, using the effective annual interest rate) of solving this type of PVSS problem, what is the present value of $3,000 due 6 years hence with a discount rate of 12 percent applied monthly? First determine that the effective annual rate is 12.6825 percent. Then substitute it for the 12 percent nominal rate and solve for PV, which is $1,465.49 (rounded).

Notice that the methodology for computing the FVSS or the PVSS when compounding or discounting is more frequent than annually is the same as when it is annually, except that the effective annual interest rate must be determined and substituted for the nominal rate.

HP-10BII: Keystrokes for Computing the FVSS When Using an EAR	
Keystrokes	***Explanation***
, C ALL	clearing memory
500, +/-, PV	present value
6.1364, I/YR	effective annual rate
3, N	number of periods (years)
FV	597.8098 displayed

HP-10BII: Keystrokes for Computing the EAR When Discounting Is Monthly	
Keystrokes	***Explanation***
, C ALL	clearing memory
, DISP, 4	to set at 4 decimal precision
12, , NOM%	to enter the nominal rate
12, , P/YR	to adjust for monthly discounting
, EFF%	12.6825 displayed as the EAR
1, , P/YR, C	reset for one payment period per year

HP-10BII: Keystrokes for Computing the PVSS When Using an EAR	
Keystrokes	***Explanation***
, C ALL	clearing memory
3000, FV	future value
12.6825, I/YR	effective annual rate
6, N	number of periods (years)
PV	-1,465.4885 displayed

Adjusting the Nominal Annual Interest Rate and the Number of Periods

The second basic way of solving FVSS and PVSS problems involves using the nominal annual rate rather than the effective rate. In this approach, two adjustments must be made. First, the problem's nominal annual interest rate must be divided by the number of compounding or discounting periods per year. This reflects the fact that only a fraction of the annual rate will be applied each time compounding or discounting occurs during the year. Second, the number of years in the problem must be multiplied by the number of compounding or discounting periods per year. This reflects the total number of times that compounding or discounting of the fractional annual rate occurs. Note that the number (that is, the number of compounding or discounting periods per year) that is divided into the nominal annual interest rate and the number (that is, the number of compounding or discounting periods per year) that is used to multiply the number of years in the problem are always the same (2 for semiannual compounding or discounting, 4 for quarterly, 12 for monthly, 52 for weekly, and 365 for daily).

To illustrate this approach (that is, adjusting the nominal annual interest rate and the number of periods) of solving this type of FVSS problem, let us return to the problem of determining the amount to which $500 will grow in 3 years at 6 percent interest compounded quarterly. Using your HP-10BII to solve the problem, first clear its memory and enter 500, +/-, and PV for the present value. Next make the first of the two adjustments (that is, divide the nominal interest rate of 6 percent by the compounding frequency of 4) by entering 6, ÷, 4, =, I/YR for the quarterly interest rate. Then make the second adjustment (that is, multiply 3 years by the compounding frequency of 4) by entering 3, ×, 4, =, N for the number of compounding periods. Then solve the problem by entering FV. The answer displayed is $597.81 (rounded).

To illustrate this approach (that is, adjusting the nominal annual interest rate and the number of periods) of solving this type of PVSS problem, let us return to the problem of determining the present value of $3,000 due in 6 years hence with a discount rate of 12 percent applied monthly. Using your HP-10BII to solve the problem, first clear its memory and enter 3000 and FV for the future value. Next make the first of the two adjustments (that is, divide the nominal interest rate of 12 percent by the discounting frequency of 12) by entering 12, ÷ , 12, =, I/YR for the monthly interest rate. Then make the second adjustment (that is, multiply 6 years by the discount frequency of 12) by entering 6, ×, 12, =, N for the number of discounting periods. Then solve the problem by entering PV. The answer displayed is $1,465.49 (rounded).

HP-10BII: Keystrokes for Computing the FVSS When Adjusting the Nominal Rate and the Number of Periods

Keystrokes	***Explanation***
, C ALL	clearing memory
500, +/-, PV	present value
6,÷ , 4, =, I/YR	quarterly interest rate
3, ×, 4, =, N	number of compounding periods
FV	597.8091 displayed

When the solution to this type of problem is a value for the interest rate (I/YR) or the number of years (N) (rather than the FVSS or the PVSS), it must be remembered that these are periodic values. In other words, if the compounding or discounting frequency is other than annual, the solution will also be an other-than-annual value. To convert the solution to an annual basis, it must be adjusted by a factor. For example, in the case of monthly compounding, the factor is 12. Therefore, if the number of compounding periods is 24, the number of years (N) during which compounding takes place is 2 (that is, 24 ÷ 12 = 2).

HP-10BII: Keystrokes for Computing the PVSS by Adjusting Payments per Year [P/YR]

Keystrokes	***Explanation***
[shift key], C ALL	clearing memory
3000, FV	future value
12 [shift key] P/YR	Sets P/YR to 12
12 I/YR	Inputs 12% as nominal interest rate
6, [shift key] x P/YR	Sets N to 72
PV	-1,465.4883 displayed

Likewise, in the case of quarterly compounding, the factor is 4. Therefore, if the number of compounding periods is 8 and you are solving for the interest rate (I/YR), the result will be a quarterly value that needs to be converted to an annual value. However, in the case of an interest rate, it is insufficient to merely multiply the quarterly rate by the factor of 4, since the result of such a multiplication would be a nominal annual rate. A more accurate result is found by converting the nominal annual rate to an effective annual rate as described earlier.

FOCUS ON ETHICS: The Impact of Inflation

The financial advisor must never lose track of the impact of inflation. If a couple planning to retire in 20 years projects their annual needs at $50,000, that projection is in the current year's dollars. If inflation is estimated at 3 percent per year, at the retirement date the inflation-adjusted needs will be $90,000! Twenty years into retirement (40 years from today), the adjusted income requirement at 3 percent annual inflation would be $163,000! Viewed another way, a $50,000 income in 40 years will have a purchasing power equal to only $15,000 in today's dollars.

Often advisors supply future projections of mutual fund or insurance cash values with specific rate-of-return assumptions. The projected accumulation usually is extremely large, and the client is lulled into a dangerous complacency because inflation is ignored.

Unfortunately, fully informing the client of the perils of inflation often (unintentionally and ironically) results in frightening the client into inaction. Regardless of this possibility, the advisor's ethical responsibility is clear. Inflation is a critical factor that must be addressed in all financial projections. Withholding this information is the same as misinforming the client.

Calculating Annuity Values When Compounding or Discounting Occurs More Often Than Once per Year

Thus far we have examined FVSS and PVSS problems in which compounding or discounting occurs more frequently than annually. A related but separate topic is the question of annuity values when compounding or discounting occurs more frequently than once per year.

Simple Annuities and Simple Annuities Due

simple annuity or simple annuity due

A *simple annuity or simple annuity due* is one in which the frequency of payments and the frequency of compounding or discounting are identical. An example of a simple annuity is a series of six quarterly deposits credited with interest quarterly, beginning 3 months from now. Likewise, an example of a simple annuity due is a series of 15 monthly payments discounted on a monthly basis, beginning immediately. All of the annuity topics discussed so far involved simple annuities or simple annuities due because both the payment frequency and the compounding or discounting frequency were identical, once per year.

The calculation of the present or future value of a simple annuity or a simple annuity due when payments are more frequent than annual involves the same tools and procedures you have already learned. The same formulas you used for computing the FVA, the FVAD, the PVA, and the PVAD can be used; you can even enter the same keystrokes on your HP-10BII—except for two adjustments. First, you need to adjust the number of payments per year by entering the number of payments per year, , and P/YR. Second, you need to enter the total number of payments in the problem, not the number of years. This is done by entering the number of years, , and x P/YR. The nominal interest rate is entered as an annual rate and not a periodic rate.

A corporate bond is a frequently encountered security that includes a simple annuity with payments occurring more frequently than annually. For example, assume that a bond provides semiannual interest payments of $40 for 10 years, beginning 6 months from now, as well as payment of the $1,000 principal sum at the end of the 10th year. If bonds with a similar degree of riskiness are yielding 11 percent, what should you pay for this bond? Using your HP-10BII, first clear its memory and set it for end-of-period payments. Next make the two adjustments noted above, namely enter 2, , and P/YR to adjust for two payments per year and semiannual discounting. Then enter 10, , and x P/YR for the total number of payments over the 10-year period. Next enter 40 and PMT for the amount of each interest payment; 1000 and FV for the maturity value of the bond; 11 and I/YR for the interest rate. Then solve for PV to determine the price you should pay for the bond, which is $820.74 (rounded). Finally, enter 1, , P/YR, and C to reset your HP-10BII for one payment period per year.

Now assume that you buy this bond for $820.74 and hold it for 3 years, at which time it is called by the issuing corporation at a call price of $1,040. What has been your annual yield to the call date? To solve this problem on your

HP-10BII, first clear its memory and set it for end-of-period payments. Then enter 2, , and P/YR (for the number of payments per year); 3, , and x P/YR (for the total number of payments up to the call); 40 and PMT (for the amount of each payment); 820.74, +/-, and PV (for the purchase price of the bond); 1040 and FV (for the call price of the bond). Then to solve for the nominal annual yield enter I/YR and the answer, 16.94 percent, will appear on the screen. Then without clearing your HP-10BII's memory, you can also solve for the effective annual yield of 17.66 percent by entering and EFF%. Finally, reset your HP-10BII for one payment period per year by entering 1, , P/YR, and C.

HP-10BII: Keystrokes for Computing the Present Value of a Bond

Keystrokes	***Explanation***
, C ALL	clearing memory
, BEG/END	only if BEGIN displayed
2, , P/YR	number of payments per year
10, , x P/YR	total number of payments
40, PMT	amount of each payment
1000, FV	value of bond at maturity
11, I/YR	interest rate
PV	-820.7443 displayed
1, , P/YR, C	reset for one payment period per year

You can also calculate N or PMT in a simple annuity or simple annuity due involving other-than-annual payments. However, the item you compute as N is the total number of payments, not the total number of years. When N is one of the known values, you should enter the keystroke of: number of years, , and xP/YR (as shown in the previous problem), which will provide you with the total number of payments. This keystroke must be entered after the keystroke that enters the number of payments per year: number of payments per year, , and P/YR (as shown in the previous problem). The item you enter or compute as PMT is the single periodic payment, not the sum of the payments per year.

HP-10BII: Keystrokes for Computing the Annual Yield of a Bond	
Keystrokes	***Explanation***
, C ALL	clearing memory
, BEG/END	only if BEGIN displayed
2, , P/YR	number of payments per year
3, , x P/YR	total number of payments
40, PMT	amount of each payment
820.74, +/-, PV	purchase price of bond
1040, FV	call price of bond
I/YR	16.9444 displayed
, EFF%	17.6622 displayed
1, , P/YR, C	reset for one payment period per year

Complex Annuities and Complex Annuities Due

complex annuity

complex annuity due

A *complex annuity* or *complex annuity due* is one in which the frequency of payments and the frequency of compounding or discounting are different. For example, a complex annuity due is a series of 14 monthly deposits that are credited with interest daily, beginning immediately. Likewise, a complex annuity is a series of 10 semiannual lease payments that are discounted on a monthly basis, beginning 6 months from now.

HP-10BII: Keystrokes for Computing the EAR When Discounting Is Weekly	
Keystrokes	***Explanation***
, C ALL	clearing memory
, DISP, 4	to set at 4 decimal precision
8, , NOM%	to enter the nominal rate
52, , P/YR	to adjust for weekly compounding
, EFF%	8.3220 displayed as the EAR
1, , P/YR, C	reset for one payment period per year

Problems involving complex annuities or complex annuities due are fairly complicated and generally beyond the scope of this textbook. However, it is a relatively easy matter to solve for the present value of a complex annuity or complex annuity due in which the payments are made annually but discounting occurs more frequently. For example, what is the present value of a 5-year annuity of $2000 if it is discounted at 8 percent weekly? Using your HP-10BII, first determine the EAR. Then substitute the EAR for the nominal rate and solve for the PVA, which in the example is $7,918.11 (rounded).

HP-10BII: Keystrokes for Computing the PVA When Using	
Keystrokes	***Explanation***
, C ALL	clearing memory
, BEG/END	only if BEGIN displayed
2000, PMT	amount of payments
5, N	number of payments
8.3220, I/YR	effective annual interest
PV	-7,918.1066 displayed as the PVA

CHAPTER SEVEN REVIEW

Key Terms and Concepts

ungrouped cash flows
grouped cash flows
Fisher Effect
net present value (NPV)
internal rate of return (IRR)
conversion of interest earnings into principal
continuous compounding
effective annual interest rate (EAR)
annual percentage rate (APR)
simple annuity or simple annuity due
complex annuity
complex annuity due

Review Questions

The answers to the review questions are in the supplement. Self-test questions and the answers to them are also in the supplement and on The American College Online.

7-1. Calculate the present value of the following payments based on 8 percent interest: [7-1]

End of Year	Amount of Payment
1	$ 800
2	800
3	800
4	6,000
5	6,000

7-2. Find the present value of Ted's vested renewal commissions discounted at 9 percent interest if he expects $30,000 next year, $20,000 in 2 years, and $10,000 per year in years 3 through 9. [7-1]

7-3. The divorce is final and you have been awarded the following alimony: $5,000 at the end of each of the next 3 years; $6,000 at the end of each of the following 5 years; and $7,000 at the end of each of the following 10 years. If you remarry, however, you receive no further alimony. Measured in terms of present value and at a discount rate of 6.5 percent, how much alimony will you relinquish if you remarry today? [7-1]

7-4. Which of the following income streams would you rather have if interest rates currently are 7 percent? [7-1]

Beginning of Year	Stream A	End of Year	Stream B
1	$2000	1	$ 0
2	2500	2	0
3	3000	3	5500
4	3000	4	5500
5	3000	5	5500

7-5. Your racehorse is sure to win $60,000 at the end of each of the next 3 years, after which you believe that the horse will be able to earn about $10,000 per year in stud fees at the end of each of 5 years. If you insist on a compound annual rate of return of at least 20 percent, what is the minimum offer you would accept for the horse today? [7-1]

7-6. Tuition at the university your daughter will be attending next year is expected to be $11,000, which includes an annual inflation assumption of 8 percent per year. You plan to set aside just enough money today that, invested at 6 percent interest, will be sufficient to pay her tuition in full at the start of each of the next 4 years. How large a capital sum is needed to accomplish this objective? [7-2]

7-7. Jerry wants to set aside a sum of money today that, together with the interest earnings, will be just enough to pay his son's four annual college tuition bills. He expects tuition to increase by 8 percent annually and that freshman tuition, due one year from now, will cost $9,000. If the college fund can be expected to earn 11 percent per year, how much money should Jerry set aside today? [7-2]

7-8. Sarah is the recipient of a trust fund that has earned 8 percent interest annually. It was originally funded several years ago with $4,000 deposited at the end of the first year. The only other deposits were $1,000 per year at the end of the 18th, 19th, and 20th years of the trust's life. How much is available in the trust at the end of the 20th year? [7-3]

7-9. To convince yourself of the wisdom of your recent decision to quit smoking (and this time you really mean it), you plan at the end of each of the next 5 years to put into a savings account earning 6 percent compound annual interest the money you would have spent on cigarettes. You anticipate that the amounts of the five deposits will be $400, $450, $500, $550, and $600. If all goes according to plan, how much will be in your account after 5 years? [7-3]

7-10. Rusty just opened an IRA account with a $1,000 deposit. If he increases the size of each annual deposit by 5 percent and if the account earns 7 percent compound annual interest, how much will be in the account at the end of 5 years? [7-4]

7-11. Your goal is to have accumulated a nest egg of $75,000 when you retire 12 years from now. You plan to make an initial deposit of $2,000 today, and on each of the following 11 anniversary dates you will deposit an amount that is 10 percent higher than the previous year's deposit. If your deposits earn 7 percent interest per year, will you reach your goal? [7-4]

7-12. Using a discount rate of 12 percent, compute the net present value of an investment that is expected to produce the following pattern of cash flows [7-5]

Beginning of Year	**Amount of Cash Flow**
1	$10,000 outflow
2	3,000 outflow
3	4,000 inflow
4	5,000 inflow
5	6,000 inflow
6	7,000 inflow

7-13. You have just made a loan of $85,000 to a friend who has agreed to the following repayment schedule:

End of Year	Amount of Payment
1	$ 5,000
2	10,000
3	0
4	0
5	100,000

a. What is the net present value of this loan if 10 percent is your minimum acceptable compound annual rate of return? [7-5]
b. What is the internal rate of return of this loan? [7-6]

7-14. Diane is considering an investment that will involve an initial cash outlay of $1,000 and expected cash inflows of $230 at the end of each of the next 6 years. If the investment performs exactly as expected, what will be its internal rate of return? [7-6]

7-15. Tax reform legislation notwithstanding, you have located an exotic tax-sheltered investment opportunity. For an initial outlay of $50,000 and an additional $10,000 5 years from now, you will receive the following income stream

Timing of Flow	Income Stream
End of years 1 through 4	$7,500
End of years 6 through 10	8,000

What is the internal rate of return on this investment? [7-6]

7-16. You are debating whether to invest $100,000 in a piece of equipment that will produce the following cost savings (after tax) for your business:

Timing of Savings	Amount of Savings
End of years 1 through 3	$30,000
End of year 4	(10,000)
End of years 5 through 7	20,000

a. What is the internal rate of return on this investment? [7-6]
b. What is the net present value of this investment at an 8 percent discount rate? [7-5]

7-17. Calculate the effective annual interest rate if a 12 percent nominal annual rate is compounded
a. annually [7-7]
b. quarterly [7-7]
c. weekly [7-7]

7-18. Bob purchases a $500 certificate of deposit that pays 6 percent interest compounded semiannually. What will be the value of the CD in 5 years? [7-7]

7-19. Find the present value of a 2-year annuity based on 8 percent annual interest for
a. quarterly payments of $300 each [7-7]
b. semiannual payments of $600 each [7-7]

7-20. Harold is borrowing $1,000 from his father to repair his car. The loan is to be repaid monthly over one year, beginning one month from now, with monthly compounding based on 12 percent annual interest. Calculate the level monthly payment. [7-7]

7-21. Rita plans to save for a trip that she expects will cost $2,000 when she takes it one year from now. Calculate how much she must save each month, beginning one month from now, to meet her objective if she gets monthly compounding based on a 12 percent nominal annual interest rate. [7-7]

7-22. Calculate the present value of an income stream of $100 per month, beginning one month from now and continuing for 2½ years, discounted at 12 percent compounded monthly. [7-7]

7-23. Compute the effective annual interest rate when a nominal annual rate of 18 percent is compounded
a. semiannually [7-7]
b. quarterly [7-7]
c. monthly [7-7]

7-24. Where should you put your money: in a certificate of deposit that will earn 9.75 percent compounded daily (365 days), or in one that will earn 10 percent compounded semiannually? [7-7]

7-25. a. a. Show which is the larger amount: the future value of a 10-year, $2,000 annual annuity growing at a nominal annual interest rate of 5 percent compounded weekly or one growing at a nominal annual interest rate of 5 percent compounded monthly. [7-7]
b. Show which is the larger amount: the present value of a 6-year, $3,000 annual annuity discounted at 11 percent applied quarterly or one discounted at 11 percent applied monthly. [7-7]

7-26. Assume that you plan to save for Junior's college education by depositing $200 per month for the next 12 years in a savings account, beginning immediately. The account is expected to earn a nominal annual rate of 6 percent, compounded monthly. How much will be in the account at the end of the 12th year? [7-7]

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