6

Time Value of Money: Basic Concepts and Applications

Learning Objectives

An understanding of the material in this chapter should enable the student to

6-1. Describe several basic concepts underlying the time value of money.

6-2. Calculate the future value of a single sum, and calculate the number of periods or the interest rate in future-value-of-a-single-sum problems.

6-3. Calculate the present value of a single sum, and calculate the number of periods or the interest rate in present-value-of-a-single-sum problems.

6-4. Calculate the future value of an annuity or an annuity due, and solve sinking fund problems.

6-5. Calculate the present value of an annuity or an annuity due, and solve debt service/capital-sum-liquidation problems.

6-6. Create an amortization schedule for a level payment loan, and delineate the level payment into principal and interest components.

6-7. Solve single sum and annuity problems with five values.

This chapter discusses several basic concepts that are essential to understanding the concept of money having a time value and its application to financial planning. These concepts can be divided into those involving either present values or future values. The present value concepts are the present value of a single sum (PVSS), the present value of an annuity (PVA), and the present value of an annuity due (PVAD). Future value concepts include the future value of a single sum (FVSS), the future value of an annuity (FVA), and the future value of an annuity due (FVAD). Emphasis is placed on showing how these concepts are used to solve time value of money problems in the broader context of financial planning.

This chapter explains how to solve time-value-of-money problems by using a formula or by using factors in the formula. It then shows you how to solve the problem using a financial calculator. While most financial calculators with time-value capabilities could be used to solve time-value-of-money problems, this book explains how to solve these problems by using only an HP-10BII. If you already own another financial calculator and do not want to acquire an HP-10BII, then make sure you know how to solve the various types of time-value-of-money problems with your calculator. The answers to both the review questions and self-test questions in the supplement to this text show the keystrokes for solving time-value-of-money problems for both the HP-10BII and the HP-12C. Your calculator's instruction booklet or user's guide is a good source of information about how to use it to solve time-value problems. Keystrokes for other calculators are not currently supported in the textbook.

Although this chapter and the next one present many time-value-of-money concepts mathematically with the aid of factor tables, you should avoid becoming bogged down in the math. Instead, concentrate on recognizing the different types of problems and learning how to use an HP-10BII to solve them. With practice you should be able to roughly estimate answers before solving the problems. Learning to estimate will help you get a feel for time-value-of-money problems, and you will sometimes catch calculation errors when they vary too much from your estimate.

In those instances where the same time-value-of-money problem is solved twice, first by using a formula (with the aid of factor tables) and then by using an HP-10BII, there will be minor differences in the two answers due to rounding and decimal precision. Do not dwell on these differences as they are not important. As mentioned above, what is important in this chapter and the next one is learning how to recognize the different types of time-value-of-money problems and how to solve them with a financial calculator.

Tables showing the time value of money are readily available on the Web. We recommend searching Google for "time value of money tables" or visiting http://www.flexstudy.com/demo/demopdf/96019_appendix.pdf.

THE BASICS OF TIME VALUE

An investment requires capital today and generates the expectation that, at some point in the future, the capital will be returned along with a return on the funds invested. The riskier the investment, the higher the expected return needs to be to entice investors to invest. This risk-return trade-off is a cornerstone of finance and financial planning. Investors expect to be compensated for the risk they assume in an investment. The expected return is not always realized because if expectations are always realized, investing would be a sure thing and a sure thing does not have any risk.

opportunity cost

Money now versus money later is an easy choice when it is the same dollar amount. Offer someone a choice between receiving a $1,000 today or a $1,000 a year from now and he or she will invariably take the money today. To get the person to wait for the money there needs to be an incentive to wait. That incentive is the return he or she can expect to receive by waiting. Investors want to be compensated for the use of their money by others over time. If they wait for the money, they incur what economists call an *opportunity cost*. The opportunity cost of an activity (in this case waiting to receive the money) is the value of the lost opportunity to engage in the best alternative activity (spending or investing the money now) with the same resource (the specified sum of money). The opportunity cost is typically expressed as a percentage rate of return.

time value of money (TVM)

Conversely, most people would intuitively conclude that if they must pay out a specified sum of money, they would prefer to pay it later rather than sooner. Why? Because the longer they can delay the payment, the longer they can use the money either by spending it or investing it for their own benefit. If they pay the money early, they also incur an opportunity cost. These differences in value over time, due to opportunity costs, relate to *the time value of money (TVM)*. The time value of money reflects the idea that a dollar today is worth more than a dollar received in the future.

The Role of Interest

interest

A given sum of money due in different time periods does not have the same values, so a tool is needed in order to make the different values comparable. That tool is *interest*. Interest is a way of quantifying the opportunity cost incurred by waiting to receive money or by giving up the opportunity to delay payment.

For example, if you deposit $1,000 in a savings account and leave the funds there for one year, you expect to have more than $1,000 in the account at the end of the year. You expect the account to earn interest. By postponing your use of the money and allowing the bank to use it, you incur an opportunity cost. The bank pays interest as compensation for that loss of use.

To reverse the situation, assume a loan you took out at your bank matures in one year, at which time you are obligated to pay $10,000. If you repay the loan today, a year early, you should be required to pay less than the full $10,000. If you forgo the opportunity to delay the repayment, you should be compensated by a reduction in the amount of the repayment due.

risk-free rate

risk premium

The specific interest rate used to quantify opportunity cost consists of two components: a *risk-free rate* and a *risk premium*. At a minimum, the opportunity cost of letting someone use your money is the rate of return you could have earned by investing it in a perfectly safe investment. A reasonable measure of this minimum opportunity cost is the rate of interest available on 3-month U.S. Treasury bills. These bills are always available and, for all practical purposes, risk free.

In contrast, most situations where you allow someone else to use your money entail some risk. For example, the market value of the investment may decline. Inflation may erode the purchasing power of your principal sum. The person or organization using your funds may default on scheduled interest or principal payments. Tax laws may change lowering the after-tax return on your investment. These and other types of risk associated with letting someone else use your funds should be reflected in a risk premium, in addition to the risk-free opportunity cost of money. Theoretically, the higher the degree of risk, the greater the risk premium and, therefore, the higher the interest rate you should require.

Simple Interest Versus Compound Interest

Simple interest

compound interest

There are two ways of computing interest. *Simple interest* is computed by applying an interest rate to only the original principal. *Compound interest* is computed by applying an interest rate to the sum of the original principal and the interest credited to it in previous periods. With compound interest, interest earned is added to the principal balance and also earns interest. Interest earning interest is the reason why it is called compounding.

The difference between simple and compound interest can be demonstrated with an example. Assume $1,000 is deposited in an account that earns 6 percent simple interest per year. At the end of each year, the account will be credited with $60 of interest. At the end of 5 years, there will be $1,300 in the account (if no withdrawals have been made), as shown in *Table 6-1, Accumulation of $1,000 in 5 Years at 6 Percent Simple and Compound Interest per Year*.

If, instead, the account earns 6 percent compound interest per year, the deposit will grow to a larger amount than $1,300, as shown in *Table 6-1, Accumulation of $1,000 in 5 Years at 6 Percent Simple and Compound Interest per Year*. The account grows to $1,338.23. The extra $38.23 in the account at the end of 5 years is the result of interest earned on previous interest earnings.

Notice the difference in the annual amount by which the account grows when compound rather than simple interest is credited. The balance grows by a constant amount, $60 per year, when simple interest is credited. In the case of compound interest, the account balance grows by an increasing amount each year because the interest is paid on the principal plus interest already earned. However, the rate of growth in the compound interest case remains the same 6 percent as in the simple interest case.

Table 6-1
Accumulation of $1,000 in 5 Years at 6 Percent Simple and Compound Interest per Year

	Simple Interest			Compound Interest		
Year	Principal Sum	Interest	Ending Balance	Principal Sum	Interest	Ending Balance
1	$1,000	$60	$1,060	$1,000.00	$60.00	$1,060.00
2	$1,000	$60	$1,120	$1,060.00	$63.60	$1,123.60
3	$1,000	$60	$1,180	$1,123.60	$67.42	$1,191.02
4	$1,000	$60	$1,240	$1,191.02	$71.46	$1,262.48
5	$1,000	$60	$1,300	$1,262.48	$75.75	$1,338.23

Compound interest can have a powerful impact on future value, especially when a high interest rate or a long period of time is involved. For example, in the year 1980, the consumer price index (CPI), a commonly used measure of the inflation rate, rose by 13.5 percent over the preceding year. If that rate of inflation had continued, the same bag of groceries that cost $100 at the beginning of 1980 would have cost about $355 at the beginning of 1990 and would have risen to over $2,370 by the beginning of 2005. Based on actual CPI inflation rates experienced over that time period, the groceries would have cost $158.62 in 1990, $208.83 in 2000, and $250.83 in 2007.

Most of the day-to-day situations requiring time-value-of-money calculations involve compound interest rather than simple interest.

Compounding Versus Discounting

compounding

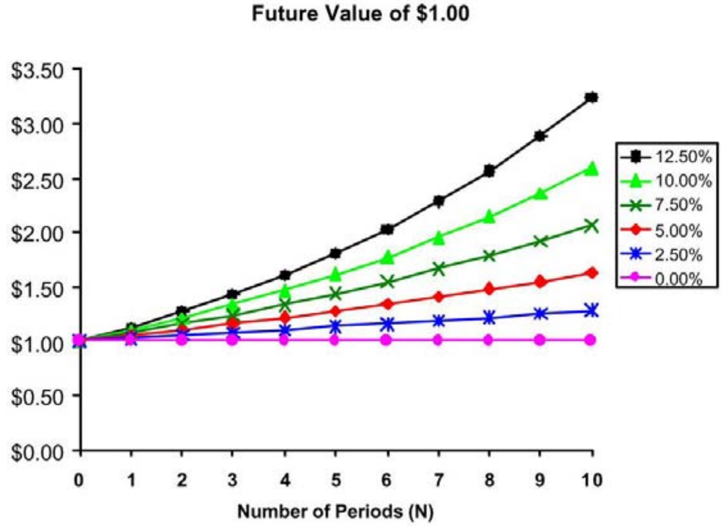
discounting

The process by which a dollar today, a present value, grows over time to a larger amount, a future value, is called *compounding*. The process by which a dollar due in the future, a future value, is reduced over time to a smaller amount today, a present value, is called *discounting*.

Figure 6-1, Compound Interest as the Link Between Present Value and Future Value shows the difference between present and future value with compound interest as the link between the two. Compounding can be viewed as a movement up the curve, while discounting can be viewed as a movement down the curve. Note also that the link between present and future value in *Figure 6-1, Compound Interest as the Link Between Present Value and Future Value* is shown as a curve (and not a straight line) to reflect the application of compound interest rather than simple interest. When compound interest is used, the future value rises each year by an increasing amount as shown by moving up the curve (or the present value declines by a decreasing amount as shown by moving down the curve).

Two major factors influence the shape of the curve in *Figure 6-1, Compound Interest as the Link Between Present Value and Future Value*. These are (1) the number of periods over which compounding/discounting occurs and (2) the interest rate used in the compounding/discounting process. Consequently, as the number of periods is increased, the difference between the present value and the future value also increases. Similarly, all other things being equal, the higher the interest rate, the steeper the slope of the curve. Thus, as the interest rate is increased, the difference between the present value and the future value also increases.

Figure 6–1 Compound Interest as the Link Between Present Value and Future Value



These lines show how time and interest rates link present and future values. Dollar amounts are reflected by the vertical axis, and the number of periods during which compounding or discounting occurs (N) is reflected on the horizontal axis. As one moves up a curve (compounding), the future value grows by increasing amounts. As one moves down a curve (discounting), the present value declines by decreasing amounts. When the interest rate is 0 percent, money has no time value and the present value is equal to the future value.

These relationships among the number of periods (N), the interest rate (i), the future value of money (FV), and the present value of money (PV) can be summarized as follows: in compounding, FV moves in the same direction as N and i (it increases as they increase); in discounting, PV moves in the opposite direction from N and i (it decreases as they increase).

Note that there are four key values in the most basic problems involving the time value of money. These values are the number of periods (N), the interest rate (i), the present value (PV), and the future value (FV). In these problems, you will be given three of the values and be called on to solve for the fourth. More complex time-value problems may include a fifth value—the value of each payment (PMT) in a series of payments.

Values in Basic TVM Problems

1. number of periods (n)
2. interest rate (i)
3. present value (PV)
4. future value (FV)

Effect of Income Taxes

nominal rate of return

Financial advisors must consider the impact of income taxes—federal, state, and local—on their time-value analyses. For example, the *nominal rate of return* realized on most investments should be adjusted downward to an after-tax return. Similarly, a borrower should adjust the nominal payment or interest rate on a loan downward to an effective payment or rate if the payments are deductible for income tax purposes. We have not explicitly factored tax considerations into the problems discussed in this chapter but have instead implicitly assumed that all values in the problems discussed are after-tax values.

Frequency of Compounding or Discounting

There is another factor in addition to the interest rate and the number of periods that affects the size of the present and future values of money. That factor is the frequency with which the interest rate is applied in the compounding or discounting process.

Throughout this chapter our focus will be on applying the interest rate once per year, which is called annual compounding or discounting. You should recognize, however, that in many cases interest rates are applied several times within a year—semiannually, monthly, daily, or even continuously.[39] For example, most coupon-paying bonds pay interest semiannually. Many certificates of deposit pay interest daily. Being able to solve time-value-of-money problems requires an understanding of how interest is paid on an investment.

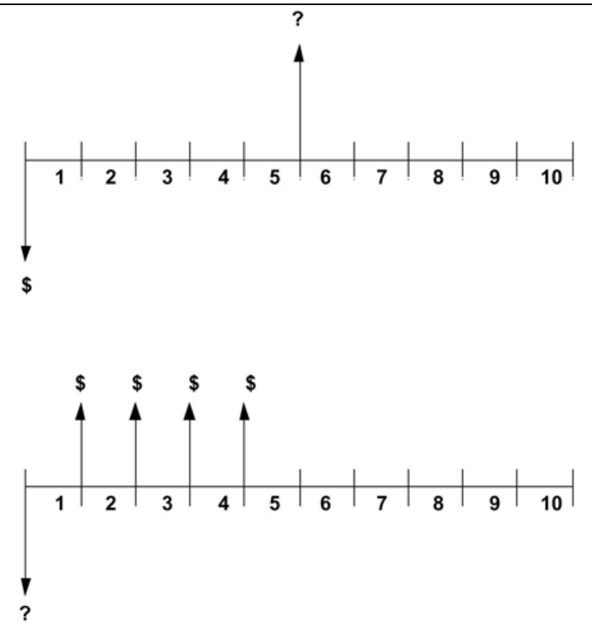
All other things being equal, the greater the frequency with which compounding or discounting occurs, the greater the effect on the growth in future values or the decline in present values. For our purposes, assume that compounding or discounting occur on an annual basis.

39. Practical applications of continuous compounding are so infrequent that the topic is not discussed further in this textbook.

Measuring the Number of Periods (Years)

Before moving on, you should keep in mind one other factor regarding the compounding or discounting process—the importance of accuracy regarding the timing of payments. Drawing time lines, such as those in *Figure 6-2, Time Lines as a Help in Counting the Number of Periods (Years) of Compounding or Discounting*, can be very helpful. The timing of known dollar values can be noted on the line with unknown dollar values denoted with question marks. In *Figure 6-2, Time Lines as a Help in Counting the Number of Periods (Years) of Compounding or Discounting*, the upper time line depicts a case in which you are to calculate the future value as of the beginning of the sixth year (which is the same as the end of the fifth year) of a deposit made at the beginning of the first year. The lower time line depicts a situation in which you are to compute the present value as of today (the beginning of year one) of a series of payments that will occur at the end of each of the next 4 years. Time lines can be constructed for all types of time-value-of-money problems.

Figure 6–2 Time Lines as a Help in Counting the Number of Periods (Years) of Compounding or Discounting



The top time line reflects a problem where a present value is deposited at the beginning of year one and you are asked to solve for the future value at the beginning of year 6, or the end of year 5. The lower time line depicts a problem where a sum of money is paid at the end of each of the next 4 years and you are asked to solve for their present value as of the beginning of year one.

FOCUS ON ETHICS: Ethics and the Time Value of Money

At first glance, a discussion of ethics may seem ill suited to the topic of the time value of money. After all, does not time value of money rely strictly on mathematical formulas without regard to ethical values? While it is true that mathematical formulas are neither moral nor immoral in and of themselves, it is also true that these formulas, improperly used, can lead to some extremely misleading results. These errant results can lead to decisions that are not in the client's best interests.

TVM formulas are dependent on interest-rate assumptions. Higher rates of return provide more optimistic results. In a competitive marketplace, it is not uncommon for projected yields to be a determining factor as the client weighs alternative courses of action. Time-value-of-money formulas do not know one yield from another. If a salesperson offers an exaggerated yield, the formula will provide an exaggerated result. It may be the decisive factor in leading the client into an inappropriate purchase.

A $10,000 tax-deferred investment that earns 8 percent per year will grow to $46,610 in 20 years. What if a financial advisor exaggerates the rate of return to 10 percent per year? The mathematical result is $67,275—a 44 percent increase in future value. Would this projection "close the sale"?

There is an old saying: "Figures don't lie, but liars figure." The ethical financial advisor uses projected rates of return that are fully justified in terms of historical experience and an analysis of future economic conditions.

Using an HP-10BII

To use your HP-10BII to solve TVM problems, turn it on by pressing the ON key in the lower left-hand corner of the keyboard. Many keys on the HP-10BII have more than one function. Keys that have second functions have orange printing on the bottom half of the key, and keys that have third functions have purple printing above the key. Pressing the orange shift key, ▭, causes the next key pressed to assume the function printed in orange. Pressing the purple shift key causes the next key pressed to assume the function printed in purple. Everywhere the shift key symbol ▭ appears in this textbook, it refers to the orange shift key, not the purple one.

We need to take care of several housekeeping tasks. Press the ▭ and C ALL keys to clear any data that may have been stored earlier in your HP-10BII's memory. It is a very good idea to get into the habit of doing this every time you turn on your HP-10BII, as "trash" left over from an earlier problem can cause an incorrect answer to the problem you are now working to solve.

HP-10BII: Clearing Memory

▭, C ALL

To set the number of decimal places displayed on your HP-10BII, press the shift key and the DISP key (which has = printed on the top half). Then press 2, 4, or 5 to specify the number of decimal places to be displayed. All calculations in this chapter will be performed at a precision of two decimal places. Therefore, before continuing, set your HP-10BII for two decimal places.[40]

HP-10BII: Changing the Number of Decimal Places Displayed

, DISP, 2 (for 2 decimal places)

Another very important housekeeping chore is to set your HP-10BII for one payment period and one compounding/discounting period per year. In chapter 7, as previously indicated, we will take up problems in which more than one payment or compounding/discounting period occurs in a year, but for now we need to keep life simple. Press 1, , and P/YR (which has PMT printed on the top half). Press and C ALL to lock this in place. ("1P - Yr" will appear briefly on the display.) Do not change this setting until you are instructed to do so in the next chapter.

HP-10BII: Setting for One Payment Period and One Compounding Period per Year

Keystrokes	***Explanation***
1, , P/YR	one compounding period per year
, C ALL	displays the number of payment periods per year

Now that the housekeeping chores are done, we will not need to discuss them again (except to remind you to clear your HP-10BII after/before every problem). We are ready to proceed to a discussion of the future value of a single sum.

40. A note about precision in calculations—when the answer is a dollar value, then rounding to the nearest penny (that is, two decimal places) is appropriate. However, calculations using interest rates or solving for an interest rate may require greater precision than two decimal places. The common unit of measure for interest rates is a basis point, which is 1/100th of a percentage point. One basis point is represented in decimal form as .0001. Twenty basis points is expressed as .0020 or 20 bps. To be able to round to the nearest basis point requires that your HP-10BII be set for five decimal precision. Therefore, there will be instances in chapter 7 when we will have you adjust your HP-10BII to four or five decimal precision.

FUTURE VALUE OF A SINGLE SUM

future value of a single sum (FVSS)

The most frequently encountered and easiest to understand application of the time-value-of-money concept involves the *future value of a single sum (FVSS)*. A single sum is an individual cash flow and not an annuity or series of cash flows. As explained earlier, determining the future value of a sum of money requires compounding, or increasing, the present value at some interest rate for a specified number of years. The most common example is the growth of a sum placed in an interest-bearing savings account. Recall, for example, that a $1,000 deposit made today (present value) will grow to $1,338.23 (future value) at the end of 5 years at 6 percent compound interest.

Future Value of a Single Sum Formula

The basic formula for computing the future value of a single sum of money, from which all other time-value formulas are derived, is the following:

FVSS formula

$$\text{FVSS} = \text{PVSS} \times (1 + i)^n$$

FVSS factor

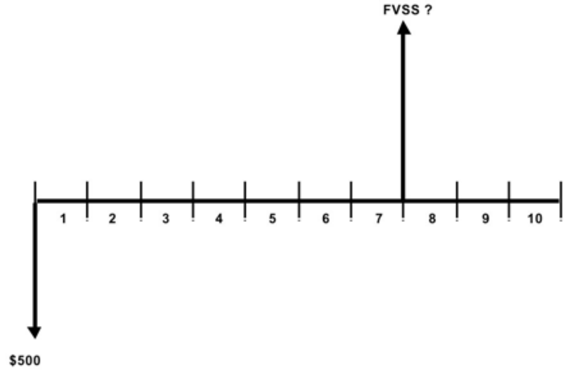
where

FVSS	=	the future value of a single sum
PVSS	=	the present value of a single sum
i	=	the compound annual interest rate expressed as a decimal
n	=	the number of years over which compounding occurs
$(1 + i)^n$	=	*FVSS factor*

When expressed in words, the FVSS formula requires you to add the interest rate (expressed as a decimal) to one and raise the sum to a power equal to the number of years over which compounding occurs. Then multiply the result by the present value of the single sum or deposit in question.

For example, assume that $500 is placed on deposit today in an account that will earn 9 percent compound annual interest. To what amount will this sum of money grow by the end of year 7? This problem is depicted on a time line in *Figure 6-3, Time Line Depiction of FVSS Problem* and, as shown in the next section, can be solved using the FVSS formula.

Figure 6–3 Time Line Depiction of FVSS Problem



This time line depicts a problem in which a known single sum, $500, is deposited today, at the beginning of year one, and you are to calculate its future value as of the end of year seven. The time line also illustrates the basic trade-off present in all time-value-of-money problems. Here the trade-off is a cash outflow today (the deposit shown below the time line) for a larger cash inflow later (the account balance shown above the time line at the end of the seventh year).

It is important, both conceptually and mathematically, to recognize that in every time-value-of-money problem there is an implicit trade-off over time of a sacrifice for a gain or a cost for a benefit. For instance, you may be willing to loan money to a friend today (a cost or cash outflow in the present) in order to be repaid a larger amount later (a benefit or larger cash inflow in the future). Throughout the time-value-of-money discussions in this chapter and the next, the nature of this trade-off will be pointed out repeatedly. For purposes of consistency when using the time lines to depict various types of TVM problems, future values and periodic cash inflows will be shown above the line, while present values and periodic cash outflows will be shown below the line.

Using the FVSS Formula

Returning to the problem at hand, the $500 placed on deposit today represents a present value. The future value to which it will grow at 9 percent compound annual interest by the end of the seventh year can be computed using the FVSS formula with the appropriate FVSS factor as follows:

$$\begin{aligned} \text{FVSS} &= \text{PVSS} \times \text{FVSS factor} \\ &= \text{PVSS} \times (1 + i)^n \\ &= \$500 \times 1.09^7 \\ &= \$500 \times 1.8280 \\ &= \$914.00 \end{aligned}$$

Computing the FV With an HP-10BII

Using your HP-10BII to calculate a FV requires that you enter the three known values in the future value problem, namely, the present value of the single sum (PV), the number of periods or years (N), and the interest rate per year (I/YR). These known values may be entered in any order, as will be explained momentarily when we work through the above problem. However, you must remember the cost-benefit trade-off found in every TVM problem that we mentioned earlier. Entering a number as a negative is accomplished through the use of the +/- key in the left-hand column of the keyboard. For purposes of consistency, we will enter or display present values as negative numbers. Later we will take up problems involving periodic cash flows in or out. Periodic outflows will also be entered or displayed as negative numbers.

Take a moment to review the top row of the keyboard. The first five keys (N, I/YR, PV, PMT, and FV) as well as the +/– key and, later, the shift key, will be used to solve various types of TVM problems.

The HP-10BII employs a cash flow sign convention. This convention requires that inflows and outflows have opposite signs. This means that when a present value is entered as a negative number, the future value calculated will be displayed as a positive number. Conversely, when a present value is entered as a positive number, the future value calculated will be displayed as a negative number. To designate a number as negative, simply press the +/- (change sign) key after pressing the key for the final digit. One approach to remembering what sign to use with a cash flow is to think of inflows as being good, so enter them as positive numbers. Outflows can be viewed as bad; consequently, enter them as negative numbers.

Returning to the example we solved using the FVSS formula, assume that you want to know the amount to which a single sum of $500 will grow in 7 years at 9 percent compound annual interest. The steps below explain how to solve the FVSS problem.

1. if you have not already done so, set your HP-10BII to display two decimal places. (We will not remind you to do this again.)
2. clear its memory by pressing and C ALL.
3. enter 500, +/–, and PV
4. press 9 and I/YR (because 9 percent is the compound annual interest rate) These two entries (that is, 9 and I/YR) are a sequence, the order of which cannot be varied when being entered.
5. press 7 and N (because 7 is the number of periods or years in the problem). These two entries (that is, 7 and N) are also a sequence, the order of which cannot be varied when being entered.

The order in which the three sequences (with each sequence representing a known value) are entered can be varied. They can be entered in any order without affecting the answer. Finally, press FV (the unknown value to be calculated) and the answer, $914.02, will appear on the display screen. This answer ($914.02) is slightly different than the answer obtained using the FVSS formula and factor tables ($914.00). However, as noted at the beginning of the chapter, this difference is due to rounding and decimal precision and can be ignored.

HP-10BII: Keystrokes for Computing the FVSS	
Keystrokes	***Explanation***
, C ALL	clearing memory
500, +/-, PV	present value
9, I/YR	interest rate
7, N	number of periods (years)
FV	914.02 displayed as FVSS

If you want to change one of the data items in the problem, you may do so without reentering all the information. For example, if you want to recalculate the same problem with an 8 percent compound annual interest rate, before clearing your HP-10BII, simply enter 8 and I/YR. Then enter FV and the new amount, $856.91, will appear on the screen.

Impact of a Change in the Interest Rate or the Number of Years

To reiterate a point made earlier in this chapter, the higher the interest rate or the greater the number of years, the larger the future value. Conversely, the lower the interest rate or the fewer the number of years, the smaller the future value. So keeping money invested longer increases its future value, as does earning a higher return on the money. Lowering the return or shortening the investment horizon has the opposite effect on money; it decreases its future value.

Rule of 72 Approximation

So far we have calculated the future value of a single sum. Being able to estimate the impact of interest on a single sum can be useful when you do not have a financial calculator handy to figure out the exact dollar value. In this situation, the Rule of 72 can be useful.

Rule of 72

The *Rule of 72* is a quick method of estimating how long it will take for a sum to double at some interest rate. The formula is

$$72 \div i = n$$

where i = the interest rate expressed as a whole number, that is, 7 percent is stated as 7

n = the number of years it will take a single sum earning i to double

For example, at an annual interest rate of 9 percent, a single sum of one dollar will double in value and reach $2 in approximately 8 years (72 ÷ 9). This value will double again and reach $4 in approximately another 8 years and double still again, reaching $8, at the end of approximately 8 more years. On the other hand, at a compound annual interest rate of 4 percent the growth of the single sum will be slower; it will take about 18 years (72 ÷ 4) for each doubling to occur.

A restatement of the formula will approximate the interest rate that will be required to double an amount within a certain time period. We can restate the formula by multiplying both sides of the equation by i/n to result in 72/n = i.

To illustrate, if you want to double an amount in 10 years, then you will need to invest at 7.2 percent (72 ÷ 10 = 7.2).

Remember that the Rule of 72 provides only an approximation and that for most purposes you will want to be more precise. The higher the interest rate used the less precise the result using the Rule of 72. Knowing how to use this rule often gives you a quick and easy way to double check your work when using your HP-10BII.

Solving for the Number of Periods (Years) or the Interest Rate With an HP-10BII

Sometimes both the future value and the present value are known and you need to solve for either the interest rate or the number of periods (years). As long as you know three of the four values you can compute the fourth. For example, if you are depositing $1,000 in an account paying 7.5 percent interest compounded annually and want to determine how long it will take for this account to reach $1,500, you would use the PVSS, FVSS, and i to compute the value of n. In contrast, if you want to determine what compound annual interest rate you must earn on $6,000 that you have available to invest today in order to have $10,000 in 6 years, you would use the PVSS, FVSS, and n to compute the value of i.

Although a formula can be used to solve for the number of periods (years) or the interest rate, using an HP-10BII is faster and easier. Therefore, with the aid of your HP-10BII, determine how long it would take for a $1,000 deposit to reach $1,500 when the account earns 7.5 percent compound annual interest. First, clear the HP-10BII's memory by pressing and C ALL. Then press 7.5 and I/YR; 1000, +/-, and PV; 1500 and FV. Finally, press the N key and the answer, 5.61 years, will be displayed.

HP-10BII: Keystrokes for Computing n	
Keystrokes	***Explanation***
, C ALL	clearing memory
7.5, I/YR	interest rate
1000, +/-, PV	present value
1500, FV	future value
N	5.61 displayed as years

Note that the PV is entered as a negative number because the $1,000 deposit in the account is a cash outflow. The FV is a positive number because the $1,500 accumulation goal for the account is a cash inflow to the depositor.

In the second problem, you want to know the interest rate you must earn on $6,000 in order to have $10,000 in 6 years. Using your HP 10BII to solve for the unknown value, that is, the interest rate, first enter and C ALL to clear the HP-10BII's memory. Then enter 6 and N; 6000, +/-, and PV; 10000 and FV. Finally, press I/YR and the answer will appear as 8.89 percent. (For both problems, you may enter the three known values, each represented by a sequence, in any order.)[41]

HP-10BII: Keystrokes for Computing i	
Keystrokes	***Explanation***
, C ALL	clearing memory
6, N	number of periods (years)
6000, +/-, PV	present value
10000, FV	future value
I/YR	8.89 displayed as interest rate

Again, the PV is entered as a negative number because investing is a cash outflow.

PRESENT VALUE OF A SINGLE SUM

present value of a single sum (PVSS)

So far we have discussed compounding, that is, accumulating a known single sum of money at a compound annual interest rate over a specified number of years to determine a future value. Now, rather than moving forward in time and compounding, we will move back in time by discounting a future value to

41. The HP-12C will always round up to the nearest whole number in solving for n. In the example above that means that the HP-12C will return a value of 6 instead of 5.61.

a present value. We will use an interest or discount rate to calculate the *present value of a single sum (PVSS)*. A single sum is an individual cash flow and not an annuity or series of cash flows.

For example, assume that in 4 years you will spend $100,000 to replace a piece of manufacturing equipment. How much should be set aside today to pay for that equipment if the account is expected to earn 10 percent compound annual interest? In a second example, assume that 5 years from now you will receive a $95,000 single-sum distribution from a trust. How much is that distribution worth in today's dollars if the appropriate discount rate is 7 percent per year? Both of these problems are depicted on time lines in Figure 6-4.

The bottom of Figure 6-4 illustrates the second problem where there is a $95,000 trust fund distribution due in 5 years. If the fund is discounted at a compound annual interest rate of 7 percent, its present value is $67,735. That is, using the PVSS formula with the appropriate PVSS factor to solve the problem,

$$\begin{aligned} PVSS &= FVSS \times [1 \div (1 + i)^n] \\ &= \$95{,}000 \times [1 \div 1.07^5] \\ &= \$95{,}000 \times .7130 \\ &= \$67{,}735.00 \end{aligned}$$

Present Value of a Single Sum Formula

We learned earlier that the FVSS can be solved using the formula

$$FVSS = PVSS \times (1 + i)^n$$

By multiplying both sides of this equation by $1 \div (1 + i)^n$, the formula can be written as

$$FVSS \times [1 \div (1 + i)^n] = PVSS$$

or alternatively

PVSS Formula

$$PVSS = FVSS \times [1 \div (1 + i)^n]$$

where

PVSS = the present value of a single sum

FVSS = the future value of a single sum

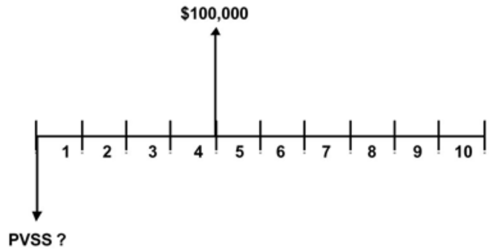
i = the compound annual interest or discount rate expressed as a decimal

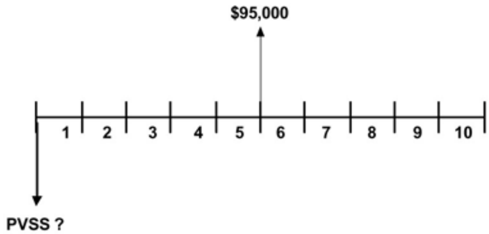
n = the number of years over which discounting occurs

PVSS factor

$$1 \div (1 + i)^n = PVSS \text{ factor}$$

Figure 6–4 Time Line Depiction of PVSS Problems





The time line on the top represents a problem in which you are asked to determine the present value of a $100,000 single sum due 4 years hence. In the time line on the bottom, the problem is to compute the PVSS when the FVSS is $95,000 due in 5 years.

Using the PVSS Formula

Using a time line, the top of Figure 6-4 illustrates the problem where $100,000 is needed in 4 years to replace some manufacturing equipment. If the funds are expected to earn 10 percent compound annual interest, you will need to set aside $68,300 today in order to have the $100,000 needed in 4 years.

Using the PVSS formula with the appropriate PVSS factor to solve the problem,

$$\begin{aligned}\text{PVSS} &= \text{FVSS} \times \text{PVSS factor}\\ &= \text{FVSS} \times [1 \div (1 + i)^n]\\ &= \$100{,}000 \times [1 \div (1.10)^4]\\ &= \$100{,}000 \times .6830\\ &= \$68{,}300.00\end{aligned}$$

That is, $68,300.00 accumulating at 10 percent compound annual interest will grow to the $100,000 needed in 4 years.

Computing the PV With an HP-10BII

However, instead of having to memorize the PVSS formula and use the PVSS factors to solve the two problems, use your HP-10BII to compute the present values. You know three of the four values. Have your HP-10BII solve for the fourth. Thus, to solve the first problem, clear your HP-10BII's memory. Then enter the three known values into your HP-10BII in any order (that is, enter the three sequences of 100000 and FV; 4 and N; 10 and I/YR in any order). Finally, press the PV key and the answer, $68,301.35, will appear on the display screen.

HP-10BII: Keystrokes for Computing the PVSS	
Keystrokes	***Explanation***
, C ALL	clearing memory
100000, FV	future value
4, N	number of periods (years)
10, I/YR	discount interest rate
PV	-68,301.35 displayed as PVSS

The negative present value on the display screen reflects that the monies are invested today, which is a cash outflow.

To solve the second problem, first clear your HP-10BII's memory and then enter in any order the three known values (that is, enter the three sequences of 95000 and FV; 5 and N; 7 and I/YR in any order). Finally, press the PV key and the answer, $67,733.69, will appear on the display screen.

HP-10BII: Keystrokes for Computing the PVSS	
Keystrokes	***Explanation***
, C ALL	clearing memory
95000, FV	future value
5, N	number of periods (years)
7, I/YR	discount interest rate
PV	-67,733.69 displayed as PVSS

Impact of a Change in the Discount (Interest) Rate or the Number of Periods (Years)

You should understand the impact on the PVSS resulting from a change in the discount rate (i) or the number of periods or years (n). The higher the discount rate, the lower the present value. The greater the number of periods or years, the lower the present value. Discounting is the stripping away of interest to arrive at the value in today's dollars. A higher discount rate or a longer investment horizon means there is more interest to be stripped away, which lowers the present value. Conversely, a lower discount rate or a shorter investment horizon means there is less interest to be stripped away, which raises the present value.

FUTURE VALUE OF AN ANNUITY OR AN ANNUITY DUE

annuity

perpetuity

Earlier in this chapter we explained how to compute the future value of a single sum credited with compound interest. Now we will build on and expand that case to deal with the calculation of the future value of a series of equal deposits or payments. For example, if $3,000 is deposited or paid into an account each year and is credited with 11 percent compound annual interest, how much will be in the account at the end of 6 years? The income stream is called an annuity. An *annuity* is a finite stream of equal periodic payments. A *perpetuity*, in contrast, is an infinite stream of equal periodic payments. Perpetuities are uncommon enough that they will not be discussed further in this textbook.

future value of an annuity (FVA)

future value of an annuity due (FVAD)

annuity due

The example describes either a *future value of an annuity (FVA)* or a *future value of an annuity due (FVAD)* problem. An annuity is a series of equal payments made at the end of each period (or year) for a specified number of periods (or years). An *annuity due* is a series of equal payments made at the beginning of each period (or year) for a specified number of periods (or years)[42]

There are many personal and business situations where money is invested periodically. Some businesses, for instance, enable their employees to invest deductions from each paycheck in U.S. government savings bonds. Many individuals deposit predetermined amounts each week or month in Christmas club or vacation club accounts at banks or credit unions. Many individuals deposit funds each year in Individual Retirement Accounts (IRAs) at banks, thrift institutions, brokerage firms, insurance companies, or mutual funds. Tax-advantaged employee retirement programs, like 401(k) plans or 403(b) tax-deferred annuity plans, enable employees to make periodic deposits that may also have matching employer contributions. Businesses may contribute sinking fund payments to accumulate money to purchase fixed assets.

Assumptions

To simplify the solution of FVA and FVAD problems, we will assume for now that the deposits or payments are made annually. Also, we will assume that the deposits or payments all earn the same rate of compound annual interest.

It is particularly important in annuity problems to accurately measure the length of time during which each deposit or payment earns compound interest. One possible assumption is that all deposits or payments are made at the

42. In some fields, such as insurance, the terms annuity and annuity due are used to refer to a series of payments, the value of which includes both compound interest and mortality factors. Such annuities are more accurately referred to as life annuities or life annuities due.

beginning of each year (an annuity due); the other is that they are all made at the end of each year (an annuity).

For example, assume that five annual deposits of $1,000 each earn 7 percent compound annual interest. At the end of the fifth year, the future value of these equal annual deposits will be $6,153.29 if they are made at the beginning of each year versus only $5,750.74 if they are made at the end of each year. The $402.55 difference between the two future values occurs because each deposit earns one more year of interest under the FVAD than under the FVA. That is, when deposits are made at the start of each year, the first deposit earns interest for 5 years rather than 4; the second deposit earns interest for 4 years rather than 3, and so on. The last deposit earns interest for one year rather than none. (See Figure 6-5 for a time line depiction.)

Using Formulas to Compute the FVA and the FVAD

A problem requiring the calculation of the future value of an annuity or annuity due can be viewed as a collection of FVSS problems. Each annuity deposit or payment can be viewed as a single sum, each of which earns compound annual interest for a different number of years. Hence, the FVA or FVAD is really the sum of a series of FVSS calculations.

To illustrate, assume that $100 is deposited at the end of each of 4 years and earns 5 percent compound annual interest. What is the total future value of these annual $100 deposits at the end of the fourth year? The first $100 deposit earns interest for 3 years (that is, from the end of year one till the end of year 4). The future value of this deposit using the FVSS formula with the appropriate FVSS factor is

$$
\begin{aligned}
\text{1st FVSS} &= \text{PVSS} \times (1+i)^n \\
&= \$100 \times (1.05)^3 \\
&= \$100 \times 1.1576 \\
&= \$115.76
\end{aligned}
$$

The future value of the second $100 deposit, which earns interest for 2 years, is

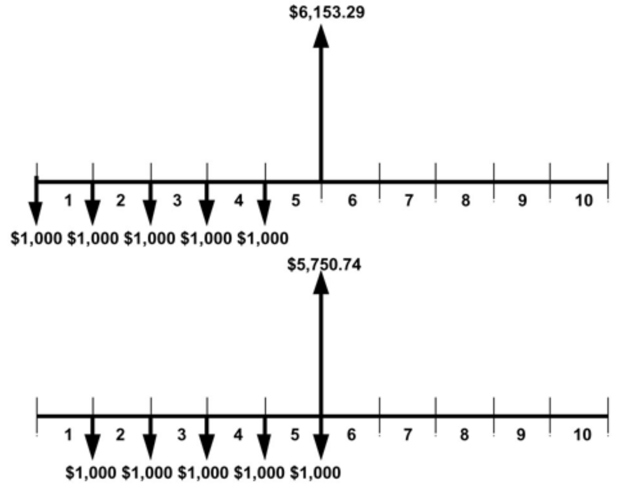
$$
\begin{aligned}
\text{2d FVSS} &= \text{PVSS} \times (1+i)^n \\
&= \$100 \times (1.05)^2 \\
&= \$100 \times 1.1025 \\
&= \$110.25
\end{aligned}
$$

The future value of the third $100 deposit, which earns interest for one year, is

$$\begin{aligned} \text{3d FVSS} &= \text{PVSS} \times (1 + i)^n \\ &= \$100 \times (1.05)^1 \\ &= \$100 \times 1.0500 \\ &= \$105.00 \end{aligned}$$

And the future value of the fourth $100 deposit is the same as its present value because it earns no interest. Thus the FVA in this problem is $431.01 (that is, $115.76 + $110.25 + $105.00 + $100.00).

Figure 6–5 Time Line Depiction of FVAD and FVA Problems



Creating a timeline is critical when solving annuity and annuity due time value of money problems. The top line depicts a problem in which $1,000 deposits are made at the beginning of each of 5 years (an annuity due) and invested at 7 percent compound annual interest. The problem is to determine the future value as of the end of the fifth year. In the lower time line, the problem is the same in all respects except that all deposits are made at the end of each of the 5 years (an annuity).

If, on the other hand, the deposits had been made at the beginning of each year, their future values would have been as follows:

1st FVSS =	$\$100 \times 1.05^4$ =	$100 × 1.2155 =	$121.55
2d FVSS =	$\$100 \times 1.05^3$ =	$100 × 1.1576 =	115.76
3d FVSS =	$\$100 \times 1.05^2$ =	$100 × 1.1025 =	110.25
4th FVSS =	$\$100 \times 1.05^1$ =	$100 × 1.0500 =	105.00
FVAD			= $452.56

For cases where the deposits are made at the end of each year, there is an alternative to the approach of summing the future values of each of the separate deposits. The same result can be achieved in one step by using the FVA formula with the appropriate FVA factor as follows:

FVA formula

$$FVA = \text{annual deposit} \times [(1 + i)^n - 1] \div i$$

where the bracketed portion is the *FVA factor.*

FVA factor

Because the bracketed portion of the equation is the FVA factor, the FVA formula can be written as follows:

$$FVA = \text{annual deposit} \times \text{FVA factor}$$

Using the FVA formula with the appropriate FVA factor to determine the future value of the deposits made at the end of each year,

$$\begin{aligned} FVA &= \text{annual deposit} \times [(1 + i)^n - 1] \div i \\ &= \$100 \times [(1 + .05)^4 - 1] \div .05 \\ &= \$100 \times 4.3101 \\ &= \$431.01 \end{aligned}$$

For cases in which the deposits are made at the beginning of each year, a modified version of the FVA formula (that is, a FVAD formula) is used. Because each deposit will be credited with one extra year of interest, it is necessary to multiply the result of the FVA formula by (1 + i). That is, if deposits are made at the beginning of each year, the FVA formula is transformed into

FVAD formula

$$FVAD = \text{FVA formula} \times (1 + i)$$

$$FVAD = \text{annual deposit} \times \{[(1 + i)^n - 1] \div i\} \times (1 + i)$$

which is the same as

$$\begin{aligned} FVAD &= (\text{annual deposit} \times \text{FVA factor}) \times (1 + i) \\ &= \$100 \times [(1.05^4 - 1) \div .05] \times (1 + .05) \\ &= \$100 \times 4.3101 \times 1.0500 \\ &= \$452.56 \end{aligned}$$

In other words, a simple way to calculate the FVAD is to calculate the FVA and then multiply the result by one plus the interest rate (1 + i).

Computing the FVA and the FVAD With an HP-10BII

As with the time-value problems discussed earlier in this chapter, an HP-10BII is a very useful tool for solving FVA and FVAD problems. Among its advantages over formulas are speed, reduced likelihood of error, and range of available values for n and i.

To use your HP-10BII to solve FVA or FVAD problems, you will be using a new key in the top row of the calculator keyboard—PMT (payment)—to reflect the fact that a series of deposits is involved rather than a single sum. Also, when solving a problem involving a series of payments or deposits, you must always remember to instruct your HP-10BII whether the payments or deposits will be made at the end of each period (FVA) or at the beginning (FVAD). This is accomplished through the use of the shift key and the BEG/END key that is in the second row.

So that your HP-10BII reflects when the payments occur in each period, press and BEG/END. If the screen displays the word BEGIN, it is set for beginning-of-year payments. If after pressing and BEG/END the screen displays no word, then it is set for end-of-year payments. If the current setting is what you want, proceed with the data entry. If the current setting is not what you want, press and BEG/END before proceeding with the data entry.

Let us return to the problem involving $100 annual deposits earning 5 percent compound annual interest over a 4-year period. If the deposits are to be made at the end of each year, check your HP-10BII to be sure that BEGIN is not displayed on the screen. If BEGIN is displayed, press and BEG/END to erase it. Then clear your HP-10BII's memory and enter 100, +/-, and PMT; 4 and N; 5 and I/YR. Finally, press FV and the answer, $431.01, will be displayed.

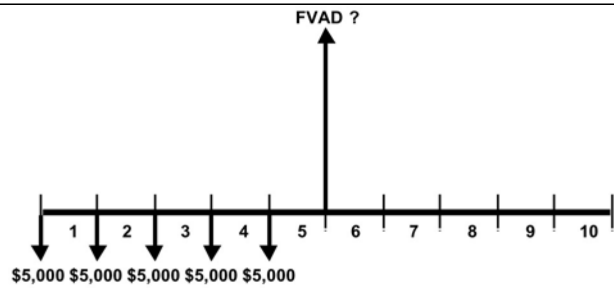
HP-10BII: Keystrokes for Computing the FVA	
Keystrokes	***Explanation***
, C ALL	clearing memory
, BEG/END	only if BEGIN displayed
100, +/-, PMT	yearly deposit
4, N	number of payments (deposits)
5, I/YR	interest rate
FV	431.01 displayed as FVA

If the $100 deposits are to be made at the beginning of each year, check your HP-10BII to be sure that BEGIN is displayed on the screen. If BEGIN is not displayed, press and BEG/END to display it. Then clear your HP-10BII's memory and enter 100, +/-, and PMT; 4 and N; 5 and I/YR. Finally, press FV and the answer, $452.56, will be displayed.

HP-10BII: Keystrokes for Computing the FVAD	
Keystrokes	***Explanation***
, C ALL	clearing memory
, BEG/END	only if BEGIN not displayed
100, +/-, PMT	yearly deposit
4, N	number of payments (deposits)
5, I/YR	interest rate
FV	452.56 displayed as FVAD

Practice is the key to learning how to use your HP-10BII so let us work through another example. Assume that a married couple deposits $5,000 today and at the start of each of the next 4 years in a savings account to accumulate funds for their young child's college education. If the account is credited with 8 percent compound annual interest, how much of a college fund will there be 5 years from now? (See *Figure 6-6, Time Line Depiction of FVAD Problem* for a time line depiction.)

Figure 6–6 Time Line Depiction of FVAD Problem



This time line depicts a 5-year annuity due. The problem is to determine the FVAD as of the end of the fifth year.

Clear your HP-10BII's memory and set it for beginning-of-year deposits (by pressing and BEG/END if BEGIN is not already displayed). Next,enter the following information to reflect the known values: 5000, +/- (as noted earlier, deposits that represent cash outflows are treated as negative numbers), and PMT; 5 and N; 8 and I/YR. Finally, press FV and the answer, $31,679.65, should appear on the display screen. However, if the deposits are to be made at the end of each of the 5 years, the keystrokes would be identical except that you would need to set your HP-10BII for end-of-year payments. Under this new assumption, the answer displayed would be $29,333.00. You can produce this solution by pressing , BEG/END and FV, rather than reentering all the information in the problem.

HP-10BII: Keystrokes for Computing the FVAD	
Keystrokes	***Explanation***
, C ALL	clearing memory
, BEG/END	only if BEGIN not displayed
5000, +/-, PMT	annual deposits
5, N	number of payments (deposits)
8, I/YR	interest rate
FV	31,679.65 displayed as FVAD

Keystrokes for Computing the FVA Without Reentering All the Information	
FVAD	31,679.65 currently displayed
Keystrokes	***Explanation***
, BEG/END	switches to end of year
FV	29,333.00 displayed as FVA

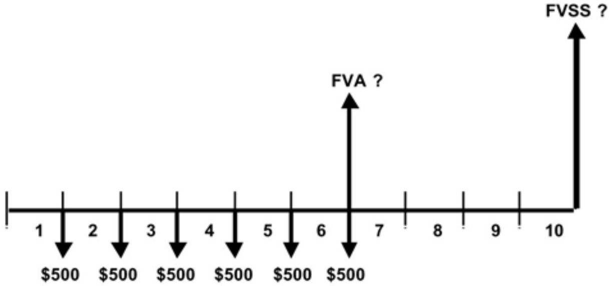
Alternatively you can start over.

HP-10BII: Keystrokes for Computing the FVA	
Keystrokes	***Explanation***
, C ALL	clearing memory
, BEG/END	only if BEGIN displayed
5000, +/-, PMT	annual deposits
5, N	number of payments (deposits)
8, I/YR	interest rate
FV	29,333.00 displayed as FVA
Keystrokes for Computing the FVAD Without Reentering All the Information	
FVA	29,333.00 currently displayed
Keystrokes	***Explanation***
, BEG/END	switches to beginning of year
FV	31,679.65 displayed as FVAD

When the Number of Periods (Years) Over Which Compounding Occurs Exceeds the Number of Periods (Years) in Which Deposits Are Made

Sometimes a problem is encountered in which the number of periods (years) during which compounding occurs exceeds the number of periods (years) during which deposits are made. For example, assume that $500 is to be deposited at the end of each of the next 6 years in an account earning 8 percent compound annual interest. How much will be in the account at the end of 10 years? (See *Figure 6-7, Time Line Depiction of a Problem Where the Number of Compounding Periods Exceeds the Number of Deposits* for a time line depiction.)

Figure 6–7 Time Line Depiction of a Problem Where the Number of Compounding Periods Exceeds the Number of Deposits



In this time line, level payments are made at the end of each of 6 years. The future value of that annuity as of the end of the sixth year is then left as a single sum to accumulate until the end of the tenth year.

This type of problem can be solved by dividing it into two separate parts and using the answer from solving the first part as an input in solving the second part. The first part of the problem involves computing the FVA; the second part of the problem involves treating the FVA as the PVSS and computing its FVSS. While the procedures for calculating the FVA and the FVSS have already been explained, solving this two-part problem not only provides an example of solving TVM problems by breaking them down into their component parts, it also provides a review of these two important future value concepts.

As previously demonstrated, the FVA can be calculated using either the FVA formula or an HP-10BII. However, because the HP-10BII is much more efficient than using the formula, it will be the only method explained here.

To solve this two-part problem using your HP-10BII, first set it for end-of-year deposits. Then find the FVA of six deposits of $500 each at 8 percent interest, which is $3,667.96. Next, clear your HP-10BII's memory and enter this amount (that is, $3,667.96) as the present value of a single sum deposited for 4 years at 8 percent compound annual interest. Solve for FV, which is $4,990.22.

HP-10BII: Keystrokes for Solving the Two-Part FV Problem	
Part 1 Keystrokes	***Explanation***
, C ALL	clearing memory
, BEG/END	only if BEGIN displayed
500, +/-, PMT	yearly deposits
6, N	number of payments (deposits)
8, I/YR	interest rate
FV	3,667.96 displayed as FVA
Part 2 Keystrokes	***Explanation***
, C ALL	clearing memory
3,667.96, +/-, PV	present value
4, N	number of periods (years)
8, I/YR	interest rate
FV	4,990.22 displayed as FVSS

Since the PV in part two of the problem was input as a negative number, your HP-10BII returns a positive number for the FVSS solution.

Solving Sinking Fund Problems With an HP-10BII

sinking fund

So far in this section we have learned how to compute the future value of a series of annual payments or deposits when the number of payments, the rate of interest, and the amount of each payment are known. Sometimes, however, the amount of the annual payment or deposit is the unknown value. This situation occurs in a *sinking fund* problem. For example, assume that a company wishes to accumulate $10,000,000 by the end of 3 years in order to retire an outstanding mortgage bond issue. The company plans to make three annual deposits into a sinking fund that will earn 9 percent compound annual interest. How large must each of the deposits be in order to reach the target amount in 3 years?

The answer to a sinking fund problem can be found by rearranging the FVA or FVAD formula and solving for the amount of the deposit. However, because the use of an HP-10BII is much more efficient than using formulas, only the HP-10BII method will be explained here.

Returning to the sinking fund problem, it is first necessary to decide whether the deposits should be made at the beginning of each year or at the end of each year. Of the two possibilities, it makes more sense to expect that the sinking fund deposits will be made at the end of each year. If they were made at the beginning of each year, the company would immediately upon receiving the $10 million bond proceeds have to make a deposit into the sinking fund.

To solve for the amount of the sinking fund deposits when they are made at the end of each year, set your HP-10BII accordingly and then clear its memory. Proceed by entering 10000000 and FV; 9 and I/YR; 3 and N. Then solve for the amount of each deposit by pressing PMT and the answer, $3,050,547.57, will be displayed.

HP-10BII: Keystrokes for Computing End-of-Year Sinking Fund Payments or Deposits	
Keystrokes	***Explanation***
, C ALL	clearing memory
, BEG/END	only if BEGIN displayed
10000000, FV	future value
9, I/YR	interest rate
3, N	number of payments (deposits)
PMT	-3,050,547.57 displayed

Switching from end-of-year payments or deposits (an annuity) to beginning-of-year payments or deposits (an annuity due) does not require that you clear your HP-10BII's memory and input all the known values again. Instead, press and BEG/END (so that BEGIN is displayed on the screen). Then press PMT to solve for the amount of a beginning-of-year deposit, which is $2,798,667.50.

If you are not able to switch from end-of-year payments or deposits to solve for beginning-of-year payments or deposits, then you must start over and input all the known values. First, however, set your HP-10BII for beginning-of-year payments or deposits and clear its memory. Then proceed by entering 10000000 and FV; 9 and I/YR; 3 and N. Then solve for the amount of each deposit by pressing PMT and the answer, $2,798,667.50, will be displayed.

The HP-10BII solutions for PMT are negative numbers reflecting the fact that the payments or deposits are viewed as cash outflows. These payments or deposits are necessary in order to reach the target $10 million, which is considered a cash inflow, at the end of year 3.

HP-10BII: Keystrokes for Computing Beginning-of-Year Sinking Fund Payments or Deposits

Keystrokes	***Explanation***
, C ALL	clearing memory
, BEG/END	only if BEGIN not displayed
10000000, FV	future value
9, I/YR	interest rate
3, N	number of payments (deposits)
PMT	-2,798,667.50 displayed

Variations of the sinking fund problem include those in which the task is to compute n (the number of deposits that will be required) or i (the interest rate that must be earned on the deposits) in order to reach the target amount. Assume, for example, that you need to accumulate $10,000 for a dream vacation to Tahiti and that you can afford to save $1,200 per year beginning a year from now. If your savings earn 10 percent compound annual interest, how long will you have to wait before you can afford the trip? While you can use the annuity formulas to solve for the number of years, the HP-10BII method is a lot less cumbersome.

To use your HP-10BII to solve the problem, first clear its memory and set it for end-of-year payments. Then enter 10000 and FV; 10 and I/YR; 1200, +/ -, and PMT. Solve by pressing N for the number of years, which is 6.36 years. Rounding up, you discover that it will take you 7 years of making annual deposits to save enough money to pay cash for the dream vacation to Tahiti.

HP-10BII: Keystrokes for Computing the Number of Years in End-of-Year Sinking Fund Payment Problems

Keystrokes	***Explanation***
, C ALL	clearing memory
, BEG/END	only if BEGIN displayed
10000, FV	future value
10, I/YR	interest rate
1200, +/-, PMT	yearly payments
N	6.36 displayed as years

On the other hand, if you insist on waiting only 5 years before going to Tahiti with your $10,000, what compound annual interest rate must you earn on your yearly deposits? Set your HP-10BII for end-of-year payments. Then enter 10000 and FV; 1200, +/-, and PMT; 5 and N. Finally, press I/YR to obtain the shocking answer of 25.78 percent.

HP-10BII: Keystrokes for Computing the Required Return in End-of-Year Sinking Fund Payment Problems	
Keystrokes	***Explanation***
▭, C ALL	clearing memory
▭, BEG/END	only if BEGIN displayed
10000, FV	future value
1200, +/-, PMT	yearly payments
5, N	number of payments (deposits)
I/YR	25.78 displayed as interest rate

PRESENT VALUE OF AN ANNUITY OR AN ANNUITY DUE

present value of an annuity (PVA)
present value of an annuity due (PVAD)

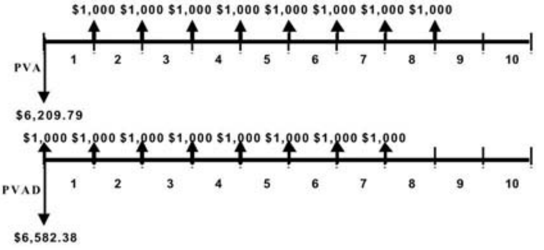
An earlier section of this chapter contained an explanation of how to calculate the present value of a single sum that is due or needed at some time in the future. This section deals with the question of how to compute the present value of a series of level future payments. This type of problem is a *present value of an annuity (PVA)* problem if the payments are made at the end of each year or a *present value of an annuity due (PVAD)* problem if the payments are made at the beginning of each year.

To illustrate this type of problem, assume that you have lent money to a business associate and expect to be repaid in eight annual payments of $1,000 each starting one year from now. How much would you be willing to sell the promissory note for today if you believe you can earn 6 percent compound annual interest on the money in an alternative investment? What is the present value of this 8-year annuity discounted at 6 percent?

Assumptions

As discussed earlier in the chapter, each payment is assumed to be made annually. This assumption will be dropped in the latter part of the next chapter. In addition, we need to specify whether the annuity payments are to be made at the end or at the beginning of each year. For example, an 8-year, $1,000 annuity discounted at 6 percent has a PVA of $6,209.79 versus a PVAD of $6,582.38. (See the time line depiction of these two types of problems in *Figure 6-8, Time Line Depiction of PVA and PVAD Problems*.)

Figure 6–8 Time Line Depiction of PVA and PVAD Problems



The upper time line depicts a case in which 8 annual payments of $1,000 are to be made beginning in one year, and the problem is to compute the PVA. The lower time line depicts an 8-year annuity due, in which $1,000 payments are to be made at the start of each year, and the problem is to compute the PVAD.

Using Formulas to Compute the PVA and the PVAD

The basic formula for computing the present value of a single sum can also be used to compute the present value of an annuity or an annuity due. All that is needed is to calculate the PVSS for each annuity payment separately and total the results.

For example, assume that as part of a divorce settlement a father has been ordered to deposit a lump sum in a trust account sufficient to provide child support payments for his son. The child support payments are to be $5,000 per year for 4 years, beginning one year from today. If the amount placed in the trust account is assumed to earn 7 percent compound annual interest, how much should the father place in the trust account today?

Using the formula for the PVSS described earlier in the chapter, the present value of each separate payment can be found. Once the present value of each separate payment has been calculated, the present value of the sum of all four payments can easily be determined.

Specifically, the present value of the first payment to be made one year from now can be calculated using the PVSS formula with the appropriate PVSS factor as follows:

$$\begin{aligned} \text{1st PVSS} &= \text{FVSS} \times [1 \div (1 + i)^n] \\ &= \$5{,}000 \times [1 \div 1.07^1] \\ &= \$5{,}000 \times .9346 \\ &= \$4{,}673.00 \end{aligned}$$

The PVSS of the second, third, and fourth payments would be calculated as follows:

2d PVSS	= $5,000 × [1 ÷ 1.07^2]	= $5,000 × .8734	= $4,367.00
3d PVSS	= $5,000 × [1 ÷ 1.07^3]	= $5,000 × .8163	= $4,081.50
4th PVSS	= $5,000 × [1 ÷ 1.07^4]	= $5,000 × .7629	= $3,814.50

The sum of these present values, the PVA, is $16,936.00 (that is, $4,673.00 + $4,367.00 + $4,081.50 + $3,814.50). This amount deposited today at 7 percent compound annual interest will be just enough to provide four annual payments of $5,000 each beginning one year from now. To verify this, examine what would happen to the account each year as that year's payment is made.

Year	Beginning Balance	Interest Earnings	Amount Withdrawn	Ending Balance
1	$16,936.00	$1,185.52	$5,000.00	$ 13,121.52
2	$13,121.52	$ 918.51	$5,000.00	$ 9,040.03
3	$ 9,040.03	$ 632.80	$5,000.00	$ 4,672.83
4	$ 4,672.83	$ 327.10	$5,000.00	($ 0.07)*

*The –.07 represents a rounding error. The ending balance must be zero. In practice the last withdrawal is adjusted to make the ending balance $0.00.

If, on the other hand, the four annual payments were to be made at the beginning of each year rather than at the end, the PVAD would be calculated by summing the four separate PVSS.

1st PVSS	= $5,000 × [1 ÷ 1.07^0]	= $5,000 × 1.0000	=	$5,000.00
2d PVSS	= $5,000 × [1 ÷ 1.07^1]	= $5,000 × .9346	=	$4,673.00
3d PVSS	= $5,000 × [1 ÷ 1.07^2]	= $5,000 × .8734	=	$4,367.00
4th PVSS	= $5,000 × [1 ÷ 1.07^3]	= $5,000 × .8163	=	$4,081.50
PVAD			=	$18,121.50

Note that a larger amount ($18,121.50 compared with $16,936.00) would have to be deposited in the trust account if the payments were to be made at the beginning of each year rather than at the end. This is because when payments are made at the beginning of each year, total interest earnings are lower. Each withdrawal to make a child support payment occurs a year earlier under a beginning-of-year assumption compared with an end-of-year assumption.

As an alternative to the approach of summing the present values of each of the separate payments, the same result can be achieved in one step by using the

PVA formula with the appropriate PVA factor for cases where the payments are made at the end of each year:

PVA formula

$$PVA = \text{annual payment} \times \{1 - [1 \div (1 + i)^n]\} \div i$$

where the bracketed portion is the *PVA factor*.

PVA factor

Because the bracketed portion of the equation is the PVA factor, the PVA formula can be written as follows:

$$PVA = \text{annual payment} \times \text{PVA factor}$$

Using the PVA formula with the appropriate PVA factor to determine the amount that needs to be deposited in the trust account in order to make the child support payments at the end of each year,

$$PVA = \$5{,}000 \times \{1 - [1 \div (1 + i)^n] \div i\}$$
$$PVA = \$5{,}000 \times \{1 - [1 \div (1.07^4)] \div .07\}$$
$$PVA = \$5{,}000 \times 3.3872$$
$$PVA = \$16{,}936.00$$

PVAD formula

For cases in which the child support payments are made at the beginning of each year, a modified version of the PVA formula (that is, a PVAD formula) is used. To reflect the fact that each child support payment will earn one less year of interest before being distributed, it is necessary to multiply the result of the PVA formula by (1 + i). That is, if the child support payments are made at the beginning of each year, the PVA formula is transformed into

$$PVAD = \text{PVA formula} \times (1 + i)$$
$$PVAD = \text{annual payment} \times [(1 - 1 \div (1 + i)^n) \div i] \times (1 + i)$$

which is the same as

$$PVAD = (\text{annual payment} \times \text{PVA factor}) \times (1 + i)$$
$$= \$5{,}000 \times [(1 - 1 \div 1.07^4) \div .07] \times 1.07$$
$$= \$5{,}000 \times 3.3872 \times 1.0700$$
$$= \$18{,}121.52$$

Computing the PVA and the PVAD With an HP-10BII

Instead of using fairly complex formulas to calculate the PVA and the PVAD, it is much easier and quicker to use an HP-10BII. An HP-10BII can simplify the task of calculating the PVA or the PVAD and is less likely to produce mistakes.

In the child support example, set your HP-10BII for end-of-year payments. Then enter 5000, +/-, and PMT; 4 and N; 7 and I/YR. Next, press PV for the answer, which is $16,936.06.

HP-10BII: Keystrokes for Computing the PVA	
Keystrokes	***Explanation***
, C ALL	clearing memory
, BEG/END	only if BEGIN displayed
5000, +/-, PMT	yearly payments
4,N	number of payments
7, I/YR	discount interest rate
PV	16,936.06 displayed as PVA

To compute the amount that needs to be in the trust account for beginning-of-year payments, simply press and BEG/END (so that BEGIN is displayed on your HP-10BII's screen). Then press PV for the answer, which is $18,121.58. There is no need to input all the known values again since they are still in your HP-10BII's memory. However, if the memory has been cleared, then the known values will have to be entered again.

HP-10BII: Keystrokes for Computing the PVAD	
Keystrokes	***Explanation***
, C ALL	clearing memory
, BEG/END	only if BEGIN not displayed
5000, +/-, PMT	yearly payments
4, N	number of payments
7, I/YR	discount interest rate
PV	18,121.58 displayed as PVAD

To sharpen your skills, work through another example. Therefore, assume that several years ago the owner of a small business borrowed $10,000 from a relative and agreed to repay the loan in 10 equal annual installments of $1,500. Today, six payments remain, the first of which is due in one year. The business owner now would like to pay off the remainder of the debt. What sum should the business owner propose to the lender as a payoff figure if money is presently earning 7 percent?

First, clear your HP-10BII's memory and set it for end-of-year payments. Then enter the following known values: 1500, +/- (because the payments are cash outflows, they are shown as negative values), and PMT (because the problem involves a series of payments, rather than a single sum); 6 and N; 7

and I/YR. The HP-10BII is now programmed to compute the PVA. Press the PV key and the answer, $7,149.81, will be displayed on the screen.

HP-10BII: Keystrokes for Computing PVA	
Keystrokes	***Explanation***
, C ALL	clearing memory
, BEG/END	only if BEGIN displayed
1500, +/-, PMT	yearly payments
6, N	number of payments
7, I/YR	discount interest rate
PV	7,149.81 displayed as PVA

If the facts of the problem are revised so that the remaining six payments are to be made at the beginning of each year, your HP-10BII first has to be set for beginning-of-year payments. Then press PV and the answer, $7,650.30, will be displayed. Like the previous example, there is no need to input all the known values again because they are still in your HP-10BII's memory. However, if you cleared your HP-10BII's memory, then you will have to re-enter the known values.

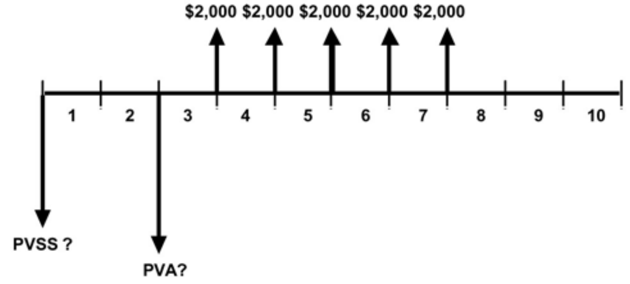
HP-10BII: Keystrokes for Computing the PVAD	
Keystrokes	***Explanation***
, C ALL	clearing memory
, BEG/END	only if BEGIN not displayed
1500, +/-, PMT	yearly payments
6,N	number of payments
7, I/YR	discount interest rate
PV	7,650.30 displayed as PVAD

When the Number of Periods (Years) Over Which Discounting Occurs Exceeds the Number of Periods (Years) in Which Payments Are Made

Sometimes a problem is encountered in which the number of periods (years) during which discounting occurs exceeds the number of periods (years) during which annuity payments are to be made. For example, discounting at a 7 percent compound annual interest rate, what is the present value of an income stream consisting of five annual payments of $2,000 each, the first of which will be made 3 years from now? (See the time line depiction of this type of problem in

Figure 6-9, Time Line Depiction of a Problem Where the Number of Discounting Periods Exceeds the Number of Payments.)

Figure 6–9 Time Line Depiction of a Problem Where the Number of Discounting Periods Exceeds the Number of Payments



This time line depicts a case in which five annual payments of $2,000 each are to be made beginning in 3 years. The problem, which is to determine the present value of the income stream, can be divided into two parts. First, determine the present value of the annuity as of the start of the annuity period, which is the beginning of year 3. (Note: It is not the beginning of year 4, even though that is when the first payment will be made.) Then compute the present value of that single sum as of today, 2 (not 3) years earlier.

deferred annuity

The way to solve this type of problem, called a *deferred annuity* problem, is to divide it into two separate parts and use the answer from solving the first part as an input in solving the second part of the problem. The first part involves computing the PVA; the second part involves treating the PVA as the FVSS and computing its PVSS.

Using an HP-10BII to solve a deferred annuity problem requires that the problem be divided into two separate parts. The first part of the problem requires you to solve for the PVA of five annual payments of $2,000 each using a 7 percent discount rate. First set your HP-10BII to end-of-year payments. Then enter 2000, +/-, and PMT; 5 and N; 7 and I/YR. Then press PV for the answer, $8,200.39, which is the PVA.

The second part of the problem requires you to use the answer, $8,200.39, from the first part (that is, the PVA) as the FVSS 2 years from now and compute its PVSS by discounting it at 7 percent. First be sure to clear your HP-10BII's memory before proceeding. Then enter 8200.39 and FV; 2 and N; 7 and I/YR. Next, press PV to calculate the answer, which is $7,162.54. In other words, $7,162.54 is the present value of an income stream consisting of five annual payments of $2,000 each, the first of which will be made 3 years from now.

HP-10BII: Keystrokes for Solving the Two-Part PV Problem	
Part 1 Keystrokes	***Explanation***
, C ALL	clearing memory
, BEG/END	only if BEGIN displayed
2000, +/-, PMT	yearly payments
5, N	number of payments
7, I/YR	discount interest rate
PV	8,200.39 displayed as PVA
Part 2 Keystrokes	***Explanation***
, C ALL	clearing memory
8200.39, FV	future value
2, N	number of periods (years)
7, I/YR	interest rate
PV	-7,162.54 displayed as PVSS

Note that the same result can also be achieved if the first part of the problem is viewed as calculating the PVAD and the second part as discounting it as a single sum for 3 additional years. (Re-examine figure 6-9 to verify that a 5-year PVA discounted for an additional 2 years is identical to a 5-year PVAD discounted for an additional 3 years.)

Solving Debt Service/Capital-Sum-Liquidation Problems With an HP-10BII

Thus far we have discussed how to compute the present value of a stream of equal annual payments when the number of payments, the discount rate of interest, and the amount of each payment are known. Sometimes, however, the amount of the annual payments (PMT) is the unknown value. A frequently encountered problem of this type involves determining the amount of the annual payments required in order to retire a debt. Likewise, a similar problem involves determining the amount of the annual withdrawals from a pool of capital in order to liquidate it over a given number of years.

debt service problem

In installment loans, like those used to finance the purchase of automobiles, each annual payment, called the debt service consists of some repayment of the principal and some payment of interest on the remaining unpaid principal. (In reality, most installment loans call for monthly payments. However, as indicated earlier in this chapter, we assumed that all periodic payments or deposits are made annually.) Given the initial size of the loan, the discount rate of interest, and the number of annual payments to be made, the problem is to compute the amount of each required payment. A *debt service problem* can be solved in a manner analogous to that used for solving sinking fund problems as explained earlier, though here we are dealing with the PVA or the PVAD rather than the FVA or the FVAD.

For example, assume that in a negligence case a jury awards $125,000 to support a 10-year-old injured child until the child reaches age 21. How much will this capital sum provide for the child each year over the next 11 years if the fund earns 8 percent compound annual interest?

Although the answer can be found by rearranging the PVA formula and solving for the amount of the payment, that procedure is complex and unwieldy when compared with using an HP-10BII. Hence, only the HP-10BII method is explained here.

You can solve the problem with your HP-10BII by setting it for beginning-of-year payments and clearing its memory. Beginning-of-year payments are selected because when a jury awards support for an injured person, that person needs some of the support money immediately. Thus, the first payment will be made now (while the child is age 10) and the last payment will be made when the child is age 20, for a total of 11 payments. The last payment should support the child until he or she reaches age 21. To proceed, enter 125000, +/-, and PV; 11 and N; 8 and I/YR into your HP-10BII. Then press PMT to determine that the amount of each support payment is $16,212.54.

HP-10BII: Keystrokes for Computing Beginning-of-Year Payments in Debt Service and Capital-Liquidation Problems	
Keystrokes	***Explanation***
, C ALL	clearing memory
, BEG/END	only if BEGIN not displayed
125000, +/-, PV	original capital amount
8, I/YR	interest rate
11, N	number of payments
PMT	16,212.54 displayed as payment

Variations of the foregoing problem are problems in which the task is to compute n, the number of years that payments will be made, or i, the interest rate that needs to be charged.

For example, assume that a company plans to borrow $200,000 to expand its fleet of delivery vans. If the financial officer believes the company can afford to make loan payments of $55,000 per year beginning one year from now, and if the lending institution is quoting an interest rate of 10.5 percent, how long will it take the company to repay the loan?

Set your HP-10BII for end-of-year payments and clear its memory. Then enter 200000 and PV; 55000, +/-, and PMT; 10.5 and I/YR. Finally, press N to find out how long it will take the company to repay the loan. The answer displayed is 4.82 years, which assumes that in the final year a fraction of a year's interest is paid as well as a fraction of a full year's loan payment.

HP-10BII: Keystrokes for Keystrokes for Computing the Required Number of End-of-Year Payments in a Debt Service or Capital-Liquidation Problem

Keystrokes	***Explanation***
, C ALL	clearing memory
, BEG/END	only if BEGIN displayed
200000, PV	original loan amount
55000, +/-, PMT	yearly loan payments
10.5, I/YR	interest rate
N	4.82 displayed as years

However, if the financial officer believes the company must have the loan repaid in 4 years but cannot afford higher annual payments, what interest rate must he or she obtain from the lending institution to accomplish this objective? Set your HP-10BII for end-of-year payments and clear its memory. Then enter 200000 and PV; 55000, +/-, PMT; 4 and N. Finally, press I/YR to find that the interest rate is 3.92 percent.

It is not realistic to think that the company has enough negotiating power with the lending institution to reduce the loan interest rate from 10.5 percent to 3.92 percent. It may have more success in negotiating a lower purchase price for the vans or extending the loan term beyond 4.82 years.

HP-10BII: Keystrokes for Computing the Interest Rate in a Debt Service or Capital-Liquidation Problem With End-of-Year Payments

Keystrokes	***Explanation***
, C ALL	clearing memory
, BEG/END	only if BEGIN displayed
200000, PV	original loan amount
55000, +/-, PMT	yearly loan payments
4, N	number of payments
I/YR	3.92 displayed as interest rate

CREATING AN AMORTIZATION SCHEDULE

amortization schedule

Another useful calculation in connection with debt service problems is to generate an *amortization schedule*. In any installment loan, as previously indicated, a portion of each payment is used to pay interest while the rest is applied to reduce the loan principal. Over the term of a loan, the portion of each payment used to pay interest declines while the portion used to reduce loan principal increases. An amortization schedule shows the portion of each payment that is applied toward interest and principal.

For example, assume that a $1,000 loan with a compound annual interest rate of 11 percent is to be repaid in four equal annual installments of $322.33 beginning one year from now. How much of each payment will be applied to interest and how much to principal?[43]

Table 6-2
Loan Amortization Schedule

Year	(1) Unpaid Balance, Beg. of Year	(2) Payment, End of Year	(3) Interest Payment i x (1)	(4) Principal Payment (2) – (3)	(5) Unpaid Balance, End of Year (1) – (4)
1	$1,000.00	$322.33	$110.00	$212.33	$787.67
2	787.67	322.33	86.64	235.69	551.98
3	551.98	322.33	60.72	261.61	290.37
4	290.37	322.33	31.94	290.39	(.02)

If you are preparing an amortization schedule manually, you should set up a worksheet with column headings as shown in above. After inserting the initial loan amount and the first year's payment, calculate the first year's interest by multiplying the 11 percent interest rate by the loan amount. This produces the figure for column (3). The balance of the payment shown in column (4) is principal and is subtracted from the initial loan amount to produce the unpaid balance at the end of the first year in column (5). This amount is also the unpaid balance at the beginning of the second year in column (1). Again, 11 percent of this amount is the second year's interest in column (3). The balance of the payment in column (4) is principal, which is subtracted from the unpaid balance at the beginning of year 2 in column (1) to produce the unpaid balance at the end

43. The procedure for creating an amortization schedule where the loan is repaid through monthly installment payments is the same as described in this chapter except that (a) the figure used as the periodic payment should be the monthly payment, (b) the figure used as the interest rate should be the annual rate divided by 12, and (c) the number used as the N should be the total number of monthly payments to be made.

of year 2 in column (5). This process is repeated through the end of the fourth and final year of the loan.

Rather than doing it manually, however, it is easier and quicker to set up an amortization schedule with your HP-10BII. Using the $1,000 loan example to demonstrate how, first set your HP-10BII for end-of-year payments and clear its memory. Then enter the loan specifics: 4 and N; 11 and I/YR; 1000 and PV. Next press PMT to obtain the yearly payment amount of $322.33. This figure is displayed as a negative number because it is a cash outflow. Then press and AMORT (located on the bottom half of the FV key) and the screen will display 1-1, indicating that the first year's payment will be broken down into its components. Next press = to display the first year's principal payment. Press = again to display the first year's interest payment. Again press = and this time the end of the first year's balance will be displayed. This process is repeated for the second-, third-, and fourth-year column breakdowns. That is, press and AMORT and then press = three times, making sure you read the screen display each time = is pressed.

HP-10BII: Calculating All Values in an Amortization	
Keystrokes	***Explanation***
, C ALL	clearing memory
, BEG/END	only if BEGIN displayed
4, N	number of payments
11, I/YR	interest rate
1000, PV	original loan amount
PMT	-322.33 displayed as payment
, AMORT	1-1 displayed
=	-212.33 first-year principal payment
=	-110.00 first-year interest payment
=	787.67 end-of-first-year balance
, AMORT	2-2 displayed
=	-235.69 second-year principal payment
=	-86.64 second-year interest payment
=	551.98 end-of-second-year balance
, AMORT	3-3 displayed
=	-261.61 third-year principal payment
=	-60.72 third-year interest payment

HP-10BII: Calculating All Values in an Amortization	
Keystrokes	***Explanation***
=	290.37 end-of-third-year balance
, AMORT	4-4 displayed
=	-290.39 fourth-year principal payment
=	-31.94 fourth-year interest payment
=	-.02 ending balance
Note: The final year principal payment usually differs from the previous year's ending balance because of a rounding error in calculating payments. Typically, the last payment is reduced so that the ending balance is zero.	

An alternative approach to setting up an amortization schedule is similar to the one just described. It provides flexibility for looking at individual years without having to look at all the years that precede it. As before, the keystrokes for the $1,000 loan example begin with setting your HP-10BII for end-of-year payments and clearing its memory. Then, as before, enter the loan specifics: 4 and N; 11 and I/YR; 1000 and PV. Next, as before, press PMT to obtain the yearly payment amount of $322.33. Then enter the number for the amortization year you want to view (such as 3), INPUT, , and AMORT and 3-3 will be displayed. Next press = three times, each time displaying a different set of information.

HP-10BII: Calculating Specific Values in an Amortization Schedule	
Keystrokes	***Explanation***
, C ALL	clearing memory
, BEG/END	only if BEGIN displayed
4, N	number of payments
11, I/YR	interest rate
1000, PV	original loan amount
PMT	-322.33 displayed as payment
3, INPUT, , AMORT	3-3 displayed
=	-261.61 third-year principal payment
=	-60.72 third-year interest payment
=	290.37 end-of-third-year balance

Your HP-10BII can easily calculate the total principal paid and total interest paid for a user-specified interval in an amortized loan. In the case of the $1,000 loan, simply enter 1000 and PV; 4 and N; 11 and I/YR. Then press PMT to calculate the yearly payment, which is displayed as -322.33 because it is a cash outflow. Next, enter the number representing the first year of the time frame being evaluated (such as 1), INPUT, the number representing the last year of the time frame (such as 4), , and AMORT. Then press = and the display will show the total principal paid during the specified time frame, which is $1,000.02 for this example. Finally, press = again and the display will show the total interest paid during the specified time frame, which in the example is $289.30.

HP-10BII: Calculating Principal Paid and Interest Paid for a User-Specified Time Interval in an Amortized Loan

Keystrokes	***Explanation***
, C ALL	clearing memory
, BEG/END	only if BEGIN displayed
1000, PV	original loan amount
4, N	number of payments
11, I/YR	interest rate
PMT	-322.33 displayed as payment
1, INPUT, 4, , AMORT	1-4 displayed
=	-1,000.02 total principal paid
=	-289.30 total interest paid

As an exercise, compute the total principal and total interest paid at the end of the third year for the 4-year loan. Your answer should be $709.63 for total principal paid and $257.36 for total interest paid.

SOLVING OTHER SINGLE SUM AND ANNUITY PROBLEMS WITH AN HP-10BII

Earlier in this chapter we noted that most TVM problems involve four values, although sometimes they involve five. In any event, the task is to solve for the unknown value whether the total number of values is four or five. For example, a problem might involve an initial deposit of $1,000 into a savings account, annual deposits of $300 at the end of years one, 2, and 3, and compound annual interest earnings of 7 percent on all the deposits. The task might be to compute the account balance at the end of the third year. Problems such as these are simply combinations of the types of problems we have already been solving. To illustrate, the preceding problem is made up of a FVSS problem ($1,000 for 3 years at 7 percent) and a FVA problem ($300 at the end of each year for 3 years at 7 percent).

To solve this problem with your HP-10BII, clear its memory and set it for end-of-year payments. Then enter 1000, +/-, and PV; 300, +/-, and PMT; 3 and N; 7 and I/YR. Then solve for FV, which should be $2,189.51.

HP-10BII: Keystrokes for Solving a FV Problem	
Keystrokes	***Explanation***
, C ALL	clearing memory
, BEG/END	only if BEGIN displayed
1000, +/-, PV	present value
300, +/-, PMT	yearly deposits
3, N	number of payments (deposits)
7, I/YR	interest rate
FV	2,189.51 displayed

A frequent situation calling for the calculation of an unknown fifth value when there are four known values is the computation of a bond's yield to maturity. For example, what is the yield to maturity of a $1,000 face amount bond, currently selling for $920, that will mature in 6 years and, in the meantime, will pay $80 of interest at the end of each year? Using your HP-10BII, first clear its memory and then set it for end-of-year payments. Then enter 1000 and FV; 920, +/-, and PV; 6 and N; 80 and PMT. Finally, press I/YR to solve for the bond's yield to maturity, which is 9.83 percent.

HP-10BII: Keystrokes for Computing Yield to Maturity of a Bond	
Keystrokes	***Explanation***
, C ALL	clearing memory
, BEG/END	only if BEGIN displayed
1000, FV	bond face amount
920, +/-, PV	current bond price
6, N	number of years (payments) to maturity
80, PMT	yearly interest payments
I/YR	9.83 displayed as yield to maturity

CHAPTER SIX REVIEW

Key Terms and Concepts

opportunity cost
time value of money (TVM)
interest
risk-free rate
risk premium
Simple interest
compound interest
compounding
discounting
nominal rate of return
future value of a single sum (FVSS)
FVSS formula
FVSS factor
Rule of 72
present value of a single sum (PVSS)
PVSS Formula
PVSS factor
annuity
perpetuity
future value of an annuity (FVA)
future value of an annuity due (FVAD)
annuity due
FVA formula
FVA factor
FVAD formula
sinking fund
present value of an annuity (PVA)
present value of an annuity due (PVAD)
PVA formula
PVA factor
PVAD formula
deferred annuity
debt service problem
amortization schedule

Review Questions

The answers to the review questions are in the supplement. Self-test questions and the answers to them are also in the supplement and on The American College Online.

6-1. You have received a bill for services rendered, and the invoice requests that you pay within 30 days. Should you pay the bill immediately on receipt or wait until the end of the 30 days? [6–1]

6-2. To what amount would $1,000 grow by the end of 3 years if it earned
a. 10 percent simple interest? [6-2]
b. 10 percent compound interest? [6-2]

6-3. Draw a time line depicting each of the following problems:
a. What amount will a deposit of $X made at the end of year one grow to by the beginning of year 7? [6-2]
b. What is the present value at the beginning of year one of a sum of $Y due to be received at the end of year 4? [6-2]

6-4. The FVSS formula for calculating the future value of a $5,000 single sum for a particular number of years and a particular interest rate can be represented as $\$5{,}000 \times 1.07^6$. What compound annual interest rate is being used, and how many years of compounding are involved? [6-2]

6-5. How is the future value of a single sum (FVSS) affected by
a. the interest rate used in the calculation? [6-2]
b. the number of years used in the calculation? [6-2]

6-6. Frank received $7,020 from his grandmother on his 10th birthday. The money was invested and has been earning 10 percent interest for the last 8 years.
a. How much money should be in Frank's account? [6-2]
b. How much interest did Frank earn from his 10 percent investment over the 8 years? [6-2]
c. How much money would be in Frank's account if the money had been deposited in a 5 percent savings account? [6-2]
d. How much interest would have been earned at the 5 percent interest rate? [6-2]
e. Was the amount of interest earned exactly twice as much when the interest rate was 10 percent rather than 5 percent? [6-2]

6-7. Lyle has an opportunity to work overseas for 5 years. He is trying to decide whether to keep his house or sell it and invest the money. If he sells the house, he will have the net sales proceeds of $100,000 to invest after paying $10,000 in selling expenses on the sale.
a. The current market value of the house is $110,000 and its value is expected to increase 4 percent each year over the next 5 years. What will the house sell for after 5 years if 4 percent growth is correct? [6-2]
b. How much will the $100,000 net sale proceeds be worth after 5 years if they are invested at 9 percent compound interest? [6-2]

6-8. At a 5 percent compound annual interest rate, approximately how long will it take for a $1,000 single sum to grow to $2,000 according to the Rule of 72? [6-2]

6-9. Approximately what rate of compound annual interest must be earned in order for a $100 single sum to grow to $300 in 10 years? [6-2]

6-10. Your personal net worth has risen in the past 4 years from $110,000 to $260,000 due to your shrewd investing. What has been the compound annual rate of growth of your net worth during this period? [6-2]

6-11. A real estate appraiser has advised you that the value of the homes in your neighborhood has been rising at a compound annual rate of about 6 percent in recent years. On the basis of this information, what is the value today of the home you bought 7 years ago for $119,500? [6-3]

6-12. According to the Rule of 72, approximately how long will it take for a sum of money to double in value if it earns a compound annual interest rate of 4 percent? [6-3]

6-13. Although you have made no deposits or withdrawals from your emergency fund savings account at the bank, the account balance has risen during the past 3 years from $15,000 to $17,613.62.
a. What has been the compound annual interest rate that the bank has been crediting to your account? [6-3]
b. At that rate, how many more years will be needed until your account balance reaches $20,000? [6-3]

6-14. The factor for calculating the present value of a single sum for a particular number of years and discount rate can be represented as

$$[1 \div (1.11)^7]$$

What discount rate is being used, and how many years of discounting are involved? [6-3]

6-15. How is the present value of a single sum (PVSS) affected by
a. the discount rate used in the calculation? [6-3]
b. the number of years used in the calculation? [6-3]

6-16. Steve invested the proceeds from the sale of his business in an investment that pays no current income to him. This investment will provide him with $60,000 when it matures in 9 years.
a. What is the present value of $60,000 when discounted at 8 percent for the 9 years? [6-3]
b. Immediately after making the investment, Steve decided to purchase another business. However, he needs the funds he just invested. His friend Paul will buy the investment at a price that yields 10 percent on his investment. How much will Paul pay for the investment? [6-3]

6-17. Assume that you owe $10,000 and that it is to be repaid in a lump sum at the end of 5 years. If the lender is willing to accept $6,000 today in full settlement of the loan, what annual rate of return (discount) is the lender effectively offering you? [6-3]

6-18. How many years will it take $10,000 to grow to $25,000 at a rate of 9 percent? [6-3]

6-19. There is an attractive piece of undeveloped land that you are considering purchasing. You think that in 5 years it will sell for $30,000. What would you pay for it today if you want to earn a compound annual rate of return of 12 percent on your investment? [6-3]

6-20. You hope to accumulate $45,000 as a down payment on a vacation home in the near future.
a. If you can set aside $38,000 now in an account that will be credited with 8 percent compound annual interest, how long will it take until you have the needed down payment? [6-3]
b. What if you can get 9 percent per year on your money? [6-3]

6-21. Draw a time line depicting a problem in which you are to calculate the future value of
a. an annuity of $100 per year for 5 years [6-4]
b. an annuity due of $200 per year for 4 years [6-4]

6-22. Using a compound annual interest rate of 12 percent, calculate the future value of
a. an annuity of $100 per year for 5 years [6-4]
b. an annuity due of $100 per year for 4 years [6-4]

6-23. Sally is contributing $2,000 each year to her individual retirement account (IRA). She makes her deposits at the end of the year. How much will she have in the IRA after
a. 10 years, if the account earns 8 percent compound interest every year? [6-4]
b. 8 years, if the account earns 7 percent compound interest every year? [6–4]

6-24. What is the future value of an annuity due of $1,500 per year for 17 years at 11.5 percent interest? [6-4]

6-25. Tracy wants to buy a house for her grandson when he finishes college 10 years from now. In the following situations, calculate how much money Tracy will have to set aside each year, beginning one year from now, in order to have $30,000 for a down payment when her grandson graduates.
a. The annual contributions earn 8 percent interest each year. [6-4]
b. Tracy assumes a 5 percent interest rate. [6-4]

6-26. You have decided that, beginning one year from now, you are going to deposit your $1,200 annual dividend check in a savings account at the credit union to build up a retirement fund. The account will be credited with 6 percent compound annual interest.
a. If you plan to retire 18 years from now, how much will be in the account at that time? [6-4]
b. If you should decide to retire 3 years earlier than that, how much will be in the account? [6-4]
c. How much would be in the account if you contributed at the beginning of each year instead of waiting a year to start contributing to the account and still retired 3 years earlier? [6-4]

6-27. The round-the-world trip you and your spouse intend to take on your 25th wedding anniversary, 6 years from now, will cost $22,000.
a. How much should you set aside each year, beginning today, to reach that objective if you can earn 9 percent compound annual interest on your money? [6-4]
b. How will the size of the annual deposit be affected if you can earn only 8 percent compound annual interest? [6-4]

6-28. You have just started a program of depositing $2,000 at the beginning of each year in an education fund account for your newborn son. How much will be in the account
a. after 11 years, if it earns 8.5 percent compound annual interest? [6-4]
b. after 13 years, if it earns 8.5 percent compound annual interest? [6-4]

6-29. Mary receives $5,000 each year from a trust. She is interested in obtaining a cash sum for the down payment on a house. Mary's cousin is willing to pay Mary the present value of the next seven trust payments (the next payment will occur one year from now) discounted at 9 percent interest if Mary will pay her the seven trust payments.
a. How much can Mary receive from her cousin for the down payment? [6-5]
b. How much can Mary receive if she convinces her cousin to discount the seven trust payments at 7 percent interest? [6-5]

6-30. A bank is willing to lend $20,000 for a home improvement loan to be repaid annually over 5 years based on 7 percent interest. What is the amount of each level payment required to repay the loan? [6-5]

6-31. A company leases an office building you own for $25,000 each year. The next rental payment is due in one year.
a. For what lump-sum amount would you today sell the next three payments if you could invest the proceeds at a 12.5 percent compound annual rate of return? [6-5]
b. If the next rental payment is due later today, what amount would you sell the payments for? [6-5]

6-32. Which would you prefer to have: $10,000 today in a lump sum or $1,000 per year for 13 years, beginning one year from now, if your opportunity cost is
a. 4 percent? [6-5]
b. 6 percent? [6-5]

6-33. The account in which you deposited your inheritance has a present balance of $48,000. If the account is credited with 10 percent compound annual interest and if you plan to withdraw $7,500 from it per year beginning one year from now, how long will it be before the balance is zero? [6-5]

6-34. Suppose that a bank will lend you $10,000 if you agree to repay $4,000 at the end of each of the next 3 years. What compound annual interest rate is the bank charging you? [6-5]

6-35. A bank is willing to lend you $15,000 to make some home improvements. The loan is to be repaid in five equal annual installments, beginning one year from now. If the interest rate on the loan is 8 percent,
a. what will be the size of the annual payment? [6-6]
b. how much of the second payment will be interest? [6-6]
c. how much of the final payment will be principal? [6-6]

6-36. If you deposit $1,100 in your bank account today, and add deposits of $600 to it at the end of each of the next 9 years, and if all your deposits earn 6 percent compound annual interest, how much will be in your account immediately after you make the last deposit? [6-7]

6-37. A bond with a $1,000 face value matures in 15 years and pays $80 in annual interest. If the bond's yield to maturity (discount rate) is 9 percent, what is the current value of the bond? what will be the size of the annual payment? [6-7]

270 South Bryn Mawr Avenue
Bryn Mawr, PA 19010
(888) AMERCOL (263-7265)
www.theamericancollege.edu